True-amplitude seismic migration: A comparison of three approaches

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ABSTRACT

Knowledge of elastic parameter (compressional and shear velocities and density) contrasts within the earth can yield knowledge of lithology changes. Elastic parameter contrasts manifest themselves on seismic records as angle-dependent reflectivity. Interpretation of angle-dependent reflectivity, or amplitude variation with offset (AVO), on unmigrated records is often hindered by the effects of common-depth-point smear, incorrectly specified geometrical spreading loss, source/receiver directivity, as well as other factors. It is possible to correct some of these problems by analyzing common-reflection-point gathers after prestack migration, provided that the migration is capable of undoing all the amplitude distortions of wave propagation between the sources and the receivers.

A migration method capable of undoing such distortions and thus producing angle-dependent reflection coefficients at analysis points in a lossless, isotropic, elastic earth is called a “true-amplitude migration.” The principles of true-amplitude migration are simple enough to allow several methods to be considered as “true-amplitude.” I consider three such migration methods in this paper: one associated with Berkhout, Wapenaar, and co-workers at Delft University; one associated with Bleistein, Cohen, and co-workers at Colorado School of Mines and, more recently, Hubral and co-workers at Karlsruhe University; and a third introduced by Tarantola and developed internationally by many workers. These methods differ significantly in their derivations, as well as their implementation and applicability. However, they share some fundamental similarities, including some fundamental limitations. I present and compare summaries of the three methods from a unified perspective. The objective of this comparison is to point out the similarities of these methods, as well as their relative strengths and weaknesses.

INTRODUCTION

“True-amplitude” processing aims to preserve the amplitude and phase information of recorded seismic data for interpretation, within a certain set of assumptions called a model. Ideally, all seismic processing should be “true-amplitude,” but very little actually is. Much seismic processing is designed either to remove “noise” (usually, signal that falls outside the model—multiple reflections, for example, do not fit the model for many processes) or to restore, by scaling, amplitude information lost in a previous step. Further, much processing, based on unrealistic models, introduces amplitude distortion. For example, 2-D seismic migration assumes effectively that earth properties do not vary in the y-direction, perpendicular to the \((x,z)\) image plane; that is, 2-D migration processes a seismic record as if the record contains only reflections from the \((x,z)\) plane. If earth structure varies significantly in the \(y\)-direction, as it often does, a 2-D migrated image will be distorted seriously. More subtly, migration methods based on a 2-D wave equation also effectively assume that seismic energy radiates from line sources, where the sources are infinitely long in the \(y\)-direction, and is recorded by point receivers. This assumption is incorrect—even though seismic sources have a small areal extent, they are clearly better approximated by points, with no \(y\)-extent, than by lines. The effect of using a 2-D wave-equation migration on actual seismic data is to introduce amplitude and phase errors into the reconstruction of reflectors in a 3-D earth, even an earth with no \(y\)-variations. Often the user is unaware of this fact, especially if the amplitude and phase behavior of the migration operator have been calibrated on 2-D synthetic data. As a consequence, the user will be unaware of, or unable to explain, errors in structure and amplitudes caused by the migration. Some migration methods, such as those in two and a half dimensions, assume that although earth properties do not
vary in the $y$-direction, sources are points rather than lines. When processing a single seismic line, using a 2.5-D migration is therefore inherently preferable to using a 2-D migration. In general, an awareness of the underlying model can help in determining whether a process can be considered to be “true-amplitude.”

“True-amplitude” migration, or migration/inversion, is the subject of this paper. The model for the true-amplitude migration methods to be discussed is a lossless, isotropic, elastic Earth that causes only primary reflections and whose properties (compressional and shear velocities and density) do not vary in the $y$-direction. Further, true-amplitude migration assumes that the source signature and the source/receiver directivity effects for a particular seismic survey are known. Of course, migration based on this model will fail to treat amplitudes correctly when the true medium—the Earth—is lossy, anisotropic, and generates significant multiple reflection energy, or when the assumed source and receiver characteristics are incorrect, but the ideas presented here constitute a starting point. Reflection data are to be processed before stack for both structural information and to provide reflection coefficients for amplitude variation with offset (AVO) or amplitude variation with incidence angle (AVA) of information.

I shall discuss three methods. The first (the Delft approach) was pioneered by Berkhout (1985) and developed by colleagues at Delft University. The second was introduced by Bleistein et al. (1987) and developed at the Center for Wave Phenomena at the Colorado School of Mines, then modified by Habral and colleagues at the University of Karlsruhe [for a summary, see Hanitzsch (1995)], and for brevity, I denote this the CWP approach. The third (the least-squares approach) was introduced by Tarantola (1984a) and developed internationally by many workers, notably LeBarron and Clayton (1988) and Beydoun and Mendes (1989). Given the similar objectives of the three methods, there are necessarily many close relationships among them. However, given also the diversity of underlying philosophies and approaches, as well as the diversity of communication styles of the workers in the field, there are also real as well as apparent differences, which have hindered an understanding of the relationships. I shall point out the relationships as well as the fundamental differences among the three methods. Perhaps an understanding of these issues can help those interested in using the best aspects of all three approaches. I shall call on as little mathematics as possible, instead focusing on the physical principles behind the sometimes formidable mathematics.

Before discussing the true-amplitude migration methods, I must emphasize some fundamental shortcomings common to all of them. These arise from deficiencies in the underlying model. First, although all the methods assume that seismic waves propagate in an elastic earth, none of them adequately accounts for losses caused by the conversion of energy from one elastic mode to another. The Delft method assumes that multicomponent seismic records have been decomposed into $P$-(compressional) and $S$-(shear) wave responses before any other processing takes place (Wapenaar et al. 1990); both the $P$ and the $S$ records can then be processed through migration in a true-amplitude sense. However, seldom in practice is there enough overlap in the frequency bandwidths of the $P$ and $S$ surface records for the decomposition to be completely reliable. Both the CWP and the least-squares methods assume essentially that no mode conversion occurs (and thus that single-component records are adequate for migration/inversion), because the smoothing applied to the (acoustic) impedance model for the purposes of computing Green’s functions by ray tracing is incompatible with mode conversions. But in practice, seismic energy does encounter interfaces in the earth, and energy does convert between $P$ and $S$ modes. Second, all the methods ignore anisotropy, which is known to affect seismic amplitudes as well as moveout. Third, all the methods fail to correct for wavefield attenuation caused by propagation through lossy material. This attenuation can often be modeled straightforwardly by an exponential loss of amplitude along the propagation path. When this is true, correcting for this loss is equally straightforward. However, it is far from straightforward to estimate the amount of loss to be corrected, and errors in this estimation will result in exponentially growing errors in the propagation operators. Fourth, until very recently, the methods have failed to account for the presence of fine detail in the background velocity model. Wapenaar et al. (1996), however, have shown the importance of this fine detail, and the effects of its neglect on migrated amplitudes.

One can raise valid, serious objections to the entire subject of true-amplitude seismic migration based on these four issues. Claerbout (1991, 108) summarizes these objections, stating, “The phrase ‘true-amplitude migration’ has questionable meaning.” On the other hand, the analysis of amplitudes on unmigrated common-midpoint (CMP) records has contributed to a great deal of exploration success. Since true-amplitude migration can be seen as a rigorous, wave-equation–based extension of AVO analysis, it might be more costly to deny the possibilities of the technology than to accept them, imperfect as they are.

**PRINCIPLES OF TRUE-AMPLITUDE MIGRATION**

The simplest illustration of true-amplitude migration is one where no migration (repositioning of reflection data from an unmigrated section to a migrated section) takes place at all. This happens when earth properties vary in only one direction (depth $z$), there is only one reflector (at depth $z_m$), and the incident waves are vertically traveling plane waves. In this trivial 1-D case, no geometrical spreading loss occurs that can complicate the calculation of amplitudes. The wavefield observed at $z = 0$ is merely a delayed version of the incident waveform, multiplied by the normal-incidence reflection coefficient:

$$ P(z = 0) = \text{Delay} \cdot R(z_m) \cdot W(z = z_m), \quad (1) $$

where $P$ is the observed wavefield, $R$ is the reflection coefficient, and $W(z_m)$ is the incident wavefield immediately above the reflector. This expression is written in the frequency domain; in the time domain, the products become convolutions. Removing the delay is the same as moving the observation depth to $z_m$:

$$ P(z = z_m) = R(z_m) \cdot W(z = z_m), \quad (2) $$

and the reflection coefficient $R$ can be found by division. If we wish to find normal-incidence reflection coefficients for reflectors beneath the shallowest one at depth $z_m$, we need to begin accounting for transmission coefficients as the wavefield passes through the shallow reflectors on its way to and from the deep reflectors. We should also account for multiple reflections, but
recall that our primaries-only model allows us to ignore them (at our peril) in processing actual seismic data.

The situation becomes slightly more complicated if the incident plane wave is not normally incident on the reflector. This case might be called one and a quarter dimensions. Now we can invert equation (1) for \( R(z_w, \theta) \), where \( \theta \) is the incidence angle of the plane wave on the reflector. A gain, the incident and reflected plane waves undergo no geometrical spreading loss, so that amplitude preservation is trivial above the shallowest reflector. For deeper reflectors, we must begin to consider not only transmission coefficients, but also Snell's law, which changes the direction of the plane wave as it passes through a velocity discontinuity encountered at a reflector. As long as \( \theta \) remains less than the critical angle at a given reflector, we can continue to find reflection coefficients for deeper reflectors.

We can gradually add complications that force us to refine our expression to be inverted. For example, one and a half dimensions is the same as one and a quarter dimensions except that the incident wavefield is a spherical wave from a point source in three dimensions; 2-D allows lateral variations of both propagation velocity and reflector structure, but the incident wavefield is a cylindrical wave from a point source in three dimensions; 2.5-D is the same as 2-D except that the incident wave is a spherical wave. For all these cases, the fundamental problem is the same, namely to deduce an expression for the angle-dependent reflection coefficient at an image point from the expression for the observed wavefield at the receiver locations. As much as they differ in detail, all the true-amplitude migration methods share this goal.

**True-amplitude Seismic Migration**

The Delft approach has been summarized in de Bruin et al. (1990), de Bruin (1992), and Berkhout and Wapenaar (1993).

For a single horizontal reflector with a constant-velocity, constant-density overburden, equation (3) is not only intuitively appealing; it also provides a rigorously correct expression for the recorded wavefield \( P \). This is an advantage over the other two approaches which, as we shall see, provide an expression for \( P \) that is only asymptotically correct. However, it is not clear that equation (3) provides an exact expression for \( P \) if the reflector is dipping, and equation (3) is not exact if the velocity or density in the overburden vary. Even so, the expression \( P = W_{up}R W_{down}S \) provides ample motivation for determining \( R \) given \( P \). The idea is to undo the effects of the source \( S \) and the downward and upward propagation \( W \)'s. To facilitate this inversion, let us replace the single-shot experiment of

true-amplitude migration methods

**True-amplitude Migration Methods**

Delft migration/inversion

The Delft approach has been summarized in de Bruin et al. (1990), de Bruin (1992), and Berkhout and Wapenaar (1993). For a single monochromatic point source, the process of downward propagation from the source location to a specific depth, reflection at the depth, and upward propagation from the reflector depth to the recording surface is modeled by the equation

\[ P = W_{up} R W_{down} S. \]

illustrated schematically in Figure 1, which is taken from de Bruin et al. (1990). In this equation, \( P \) is the wavefield recorded at the surface \( z = 0 \), and \( S \) is the source distribution at \( z = 0 \). In two dimensions, both \( P \) and \( S \) are functions of the horizontal variable \( x \) (in discrete terms, they are vectors), although \( S \), localized in space, has only one nonzero entry. The right-hand side of this equation expresses three sequential operations, to be read from right to left. \( W_{down} \) is the downward wavefield operator (discretely, matrix) that operates on \( S \), extrapolating in two or three dimensions the single frequency component of the source wavefield from the recording surface to the reflector depth. If velocity and density are constant between the upper surface and the reflector depth, \( W_{down} \) is the inverse spatial Fourier transform of Gazdag's (1978) phase shift operator \( \exp(i \Delta z \sqrt{\omega^2/c^2 - k^2}) \), where \( \Delta z \) is the extrapolation depth step, \( \omega \) is angular frequency, \( c \) is extrapolation velocity, and \( k \) is transverse wavenumber. Although \( S \) is concentrated at a single \( x \)-location, \( W_{down} S \), the wavefield from the source location at the reflector depth, is spread out over the range of \( x \)-values that comprise the range of influence of the operator \( W_{down} \).

Next, operator (discretely, matrix) \( R \) performs a reflection operation on \( W_{down} S \). At every \( x \)-location on the reflector depth, the downgoing incident energy \( W_{down} S \) is transformed by \( R \) into upgoing energy. A according to the wave equation, \( R \) performs a scattering of energy by a Huygens source at each \( x \)-location into all upward directions, so that \( R \) is not simply a diagonal operator expressing reflection at the specular direction, but rather a dense operator, with each column describing the full angular dependence of reflection at a given \( x \)-location. Finally, \( W_{up} \) operates on the upgoing reflected wavefield \( R W_{down} S \), extrapolating it to the upper surface. That is, \( W_{up} \) expresses the upward extrapolation of upgoing reflected waves. With the interpretation of the one-way wavefield operators \( W \) and the reflection operator \( R \) given here, Figure 1 is slightly misleading (remember, it is meant to be schematic)—the operators are not directed along raypaths, but act in much greater generality.

For a single horizontal reflector with a constant-velocity, constant-density overburden, equation (3) is not only intuitively appealing; it also provides a rigorously correct expression for the recorded wavefield \( P \). This is an advantage over the other two approaches which, as we shall see, provide an expression for \( P \) that is only asymptotically correct. However, it is not clear that equation (3) provides an exact expression for \( P \) if the reflector is dipping, and equation (3) is not exact if the velocity or density in the overburden vary. Even so, the expression \( P = W_{up} R W_{down} S \) provides ample motivation for determining \( R \) given \( P \). The idea is to undo the effects of the source \( S \) and the downward and upward propagation \( W \)'s. To facilitate this inversion, let us replace the single-shot experiment of
\[ P = W_{up} R W_{down} S. \] (4)

If there is one shot per recording station, then all the matrices are square, with the number of rows equal to the number of stations. If none of the shots are dead, matrix \( S \) is invertible. Multiplying both sides of equation (4) by \( W_{down}^{-1} \) on the left and \( (W_{down} S)^{-1} \) on the right yields \( R \) directly.

Remarkably simple as this inversion is, unfortunately it doesn’t work as it stands. The problem lies in inverting the \( W \)’s. As it expresses wave propagation from one depth level to another, the \( W \)’s handle both propagating and evanescent modes for a given frequency by leaving the magnitudes of the propagating modes unchanged while exponentially damping the evanescent modes. Likewise, application of \( W^{-1} \) will not change the magnitudes of the propagating modes, but it will cause the evanescent modes, and any noise superposed on them, to grow exponentially. Clearly, exponentially gaining noise in the evanescent modes is a disastrous procedure to apply on actual seismic data.

The problem of instability is easily overcome, at the price of a loss of accuracy that shows up as a slight loss of spatial resolution (Wapenaar and Berkhout, 1989, 267–270). The wavefield extrapolation operators \( W \) are complex-valued, and their complex conjugates \( W^* \) handle propagating modes exactly as their inverses \( W^{-1} \) do, while exponentially damping the evanescent modes as the \( W \)’s do. So replacing \( W^{-1} \) by \( W^* \) (the complex-conjugate transpose of \( W \)) in the inversion formula stabilizes the inversion without changing the processing of the propagating modes. However, since \( W^{-1} \) undoes the operation of \( W \), for example depropagating seismic energy to a single spatial location in the case of \( W_{down} S \), the approximation of \( W^{-1} \) by \( W^* \) must distort the depropagation to some degree, leading to a small amount of smearing in the downward continuation of the wavefields into the earth.

In practice, the Delft migration/inversion processes seismic data \( P \) first by removing effects of the sources \( S \) and then by removing the propagation effects \( W \). The operator multiplications are realized as spatial convolutions. Usually, the wavefield is extrapolated from one depth level to the next, so that the operators \( W \) are local; that is, they are responsible for downward extrapolating from one depth to the next both the downgoing source wavefield and the upgoing recorded wavefield. In doing this, the \( W \)’s cause a sample of the wavefield at an \( x \)-location at a depth to be combined (by the convolution and using the local value of velocity) from an aperture of samples at the previous depth; these convolutions, for each depth and all frequencies, constitute the bulk of the computer operations of the Delft migration/inversion. At each depth, \( W_{up} P \), the time-reversed, downward-continued recorded wavefield, is combined with \( S^{-1} W_{down} \), the time-reversed downward-continued inverse source wavefield. From equation (4), the multiplication of these operators yields \( R \) exactly, except for the estimation of \( S \), the finite length of the numerical approximations to the operators \( W \), and the difference between \( W^{-1} \) and \( W^* \).

A second stability issue arises in applying the numerical convolution expressed by the \( W \)’s. The spatial extent, or aperture,

of these convolutions is finite, and simply truncating the operators at the edge of the aperture leads to numerical instability when the convolutions are applied over many depth steps. Holberg (1988) and Hale (1991) have provided solutions to this problem that have very little adverse affect on the migrated amplitudes within the range of dips accurately imaged by the Delft method.

This method of extrapolating wavefields using operators \( W \) requires 2-D extrapolation of 2-D wavefields or 3-D extrapolation of 3-D wavefields. It cannot economically extrapolate in a 3-D sense a wavefield recorded in two dimensions, such as a single marine shot record. To accommodate such wavefields, Wapenaar et al. (1992) have proposed a preprocessing step transforming the wavefield to one with line sources, for subsequent 2-D extrapolation. This preprocessing step, which alters the amplitude and phase of the data, is strictly valid only for a horizontally layered Earth.

De Bruin et al. (1990) and de Bruin (1992) have presented successful numerical estimations of reflection amplitudes using the Delft method on synthetic 2-D reflection data from simple structures, but the method has not been extensively tested to obtain reflection amplitude information from data in structurally complicated areas.

**CWP MIGRATION/INVERSION**

Migration/inversion, as proposed originally by the CWP group and modified and extended at the University of Karlsruhe, is reviewed in detail in Hanitsch (1995). This method begins with a Green’s function representation of the recorded wavefield from a single reflector, analogous to equation (3). This representation is treated as an integral equation for the unknown reflection coefficient. Bleistein (1984, 1986) presents this expression and its subsequent simplification, described in the next two paragraphs. In three dimensions, the recorded wavefield is the integral over the reflecting surface of an expression involving the wavefield on the reflector and the Green’s function, or extrapolation operator, for propagation between the reflector and the recording surface. First, approximating the Green’s function with the ray-theoretic expression \( A e^{i\omega t} \) (where \( A \) is an amplitude, \( \omega \) is angular frequency, and \( t \) is traveltime along a raypath from a point on the reflector to the observation point), and then approximating the wavefield on the reflector by the product of the ray-theoretic incident wave and angle-dependent reflection coefficient, yield a simplified formula for the recorded wavefield. The wavefield at a receiver location is still expressed as an integral over the reflecting surface \( \Sigma \). By diffraction theory, all points on the reflecting surface (not just a single specular point) act as hydrogen sources that contribute to the field observed at a receiver. For a given source-receiver pair, and a given point on the reflector, the integrand is the product of \([\text{source strength } S(\omega)]\times[\text{downgoing wave } A e^{i\omega t} \text{ from the source to the reflector}]\times[\text{angle-dependent reflection coefficient } R] \times[\text{derivative } i\omega A e^{i\omega t} \text{ of the upgoing wave from the reflection point to the receiver}]\times[\text{an obliquity factor } O] \times[\text{Green’s function}]\). Integrating over all the reflection points, and relating the downgoing and upgoing amplitudes, phases, and obliquity, with the Delft \( W \)’s, relating the reflection coefficient with the diagonal part of the Delft \( R \), and relating the source strength with the Delft \( S \), leads to a expression for the
where the diagonal reflection operator \( R \) has contributed through \( W_{\text{down}} \). Further, the ray-theoretic \( W \)'s combine into a single operator as a product of (imaginary term \( i\omega A, A \) times (real obliquity factor \( O \)) times [complex phase term \( \exp(i\omega(t_r + t_\tau)) \]), so that the downward continuation of the wavefields from the source and the receivers is accomplished simultaneously.

According to equation (5), the recorded data at a receiver location resulting from a source excitation at a given shot location can be written as the integral over all points on the 2-D reflecting surface. As in the Delft scheme, the integral includes nonspecular as well as specular contributions, effectively depending on self-cancellation of nonspecular contributions. However, at a point on the reflector, the raypaths from the source and to the receiver determine a specific incidence angle, in contrast to the Delft approach, where all angles contribute to the operator \( R \). If the earth varies only in the \( x \)- and \( z \)-directions, the CWP approach allows a stationary phase approximation to the \( y \)-integral, reducing the surface integral to an integral over a curve in the \((x, z)\) plane.

All three approximations mentioned in the preceding two paragraphs are consistent with one another; they all assume that the wave propagation takes place in a high-frequency regime, or that the product \( \alpha L/c \gg 1 \), where \( \alpha \) is angular frequency, \( L \) is a characteristic distance (over which propagation velocity may change only slowly), and \( c \) is velocity of propagation. This condition is violated at depths within a few wavelengths of the recording surface, to the detriment of the image and the analysis of amplitudes in the near surface.

The phase term in the integral in equation (5) gives it some resemblance to a Fourier transform, and the CWP inversion approach exploits this fact. By assuming an inversion operator whose phase is the negative of the phase in equation (5) (e.g., Bleistein, 1987), one can solve for the weights to be applied when processing the recorded data for the reflection information. Deriving the inversion is every bit as complicated as deriving the integral equation for the recorded field. Rather than describing this derivation, I note only that the migration/inversion formula is produced only after the introduction of a factor [Beylkin’s (1985) determinant] that accounts for recording and processing geometry (though not for recording aperture), and after several applications of high-frequency asymptotics. The result, in 2, 2.5 or 3 dimensions, is a Kirchhoff migration of the recorded data, with different weights for common-shot, common-receiver, or common-offset processing. The CWP migration/inversion expression is complicated but reasonable. Just as the CWP integral equation for the recorded wavefield can be related to the Delft expression, a close relationship exists between the CWP formula and the classical heuristic for seismic migration (Dechtery, 1991).

CWP migration/inversion is available in 2.5-D form; as mentioned above, Delft migration/inversion relies instead on transforming the recorded wavefield into two dimensions. This gives the CWP approach an advantage over the Delft approach. On the other hand, the CWP migration formula is less “exact” than the Delft formula. Its ray-theoretic propagation operators are less accurate than the Delft’s (Gray and May, 1994), usually causing less accurate structural imaging, especially in the vicinity of wavefield caustics, and its diagonal reflection operator, with angle-dependent reflection coefficient occupying the main diagonal, is in principle less attractive than the Delft capability of recovering the full reflection operator. However, in situations where extreme complexity of geologic structure and/or propagation velocity cause visible differences in imaging accuracy between the two methods, it is doubtful whether the theoretically more accurate Delft method can account for enough complexity in its propagation operators to produce trustworthy amplitude information.

In cases of moderate velocity complexity, just strong enough to produce caustics in the wavefields (regions where raypaths cross each other, focusing a great deal of seismic energy in an infinitesimally small volume), the Delft approach might be superior to the CWP approach. The CWP method allows for ray bending but encounters difficulty in the vicinity of caustics in the wavefields for two reasons, namely phase changes in the ray-theoretic Green’s functions (which can be accounted for with some difficulty) and breakdown of the Beylkin determinant [which cannot be corrected, although recent work in Hantizsch et al. (1994) among others has circumvented the problem by avoiding the Beylkin determinant altogether]. In the Delft approach, caustics are handled automatically, as are multiple energy paths between source/receiver points and image points (Gray and May, 1994).

The stationary-phase calculation permits the use of a trick in the CWP approach that is not available in the Delft approach. The method of stationary phase (Bleistein, 1984) is an asymptotic technique for evaluating integrals with a phase (in our case, frequency times traveltime) function that is rapidly varying except near one or more (“stationary”) points. These integrations reproduce the integrand at the stationary points, weighted by some known factor. In the CWP migration/inversion, the integrand is a weight times the reflection coefficient at the image point, and the inversion produces the reflection coefficient. If, in a second application of the inversion, an extra factor of incidence angle (carried along as part of the ray calculations) is included in the migration weight function, the migration/inversion will produce the product of reflection coefficients times incidence angle. The two migration results can then be processed to yield reflection coefficient as a function of incidence angle, or AVO. Hantizsch (1995) has obtained numerical estimates of AVO and AVA from synthetic data, showing, among other effects, the effects on reflection amplitudes of migrating with an incorrect background velocity. A. Iso, Parsons (1986) and Hantizsch (1995) have shown successful AVO estimates on field data.

### Least-squares migration/inversion

Where CWP migration/inversion begins with an expression for a single reflection experiment (common-shot, common-receiver, or common-offset), least-squares migration/inversion starts from an expression for the data recorded in all the experiments comprising a seismic survey. For example, starting from Tarantola’s (1984a) least-squares expression for the recorded wavefield, which for a single experiment is identical to the
CWP Kirchhoff integral expression, leads to an inversion that resembles the CWP Kirchhoff migration. (It is possible to begin with a different expression for the wavefield. For example, Tarantola (1984b) describes a formalism beginning with the full (two-way) acoustic wave equation, leading to an inversion formula involving reverse-time migration.) However, the least-squares method interprets its expression for all the seismic data not as a linear integral equation to be inverted as exactly as possible, but rather as a general linear equation to be inverted in the least-squares sense.

The least-squares generalization of equation (5) is

\[ \mathbf{d} = \mathbf{Fm} \]  

(6)

where \( \mathbf{d} \) represents the recorded data, \( \mathbf{F} \) is the forward modeling or diffraction operator, and \( \mathbf{m} \) is the model to be determined. If a receiver records reflected energy from more than one source, then that receiver contributes more than one element to the vector \( \mathbf{d} \), which consists of all the seismic traces. The “model” might be elastic-parameter perturbations from a background level, or, as in the CWP model, it might be reflectivity caused by rapid changes in earth parameters. In contrast to equations (3) and (5), equation (6) is usually written and solved in the time domain. \( \mathbf{F} \) is a linear operator; that is, if a new diffractor is added to the model \( \mathbf{m} \), then a single new diffraction event will be added to the response \( \mathbf{d} \). Thus, \( \mathbf{F} \) does not account for multiple reflections. In fact, since \( \mathbf{F} \) is a diffraction operator, linear in diffractors, \( \mathbf{F} \) is identical to the CWP Kirchhoff modeling operator where many different sources and recording spreading are involved. In this notation, CWP migration/inversion has found, for a single reflection experiment, an analytical expression for \( \mathbf{F}^{-1} \). The CWP inversion operator assumes the ideal situation of an infinitely long recording spread, and it is applied to both sides of equation (6) to expose \( \mathbf{m} \). The purpose of the least-squares approach is, instead, to find a model \( \mathbf{m} \) (a point in the space of possible models) that minimizes the \( L_2 \) norm of \( \mathbf{d} - \mathbf{Fm} \) (LeBras and Clayton, 1988), under realistic conditions. After the spatial variables are discretized, equation (6) becomes a matrix equation. Then standard least-squares procedures (e.g., Strang, 1986, Chapt. 1) lead to the normal equations

\[ \mathbf{F}^T \mathbf{d} = \mathbf{F}^T \mathbf{Fm} \]  

(7)

(\( \mathbf{F}^T \) is the transpose of \( \mathbf{F} \)), whose solution is

\[ \mathbf{m} = (\mathbf{F}^T \mathbf{F})^{-1} \mathbf{F}^T \mathbf{d} \]  

(8)

The term \( (\mathbf{F}^T \mathbf{F})^{-1} \mathbf{F}^T \) is called the pseudo-inverse of the forward modeling operator \( \mathbf{F} \). \( \mathbf{F}^T \mathbf{F} \) can be computed from its definition \( \langle \mathbf{d}, \mathbf{Fm} \rangle = \langle \mathbf{F}^T \mathbf{d}, \mathbf{m} \rangle \), where the \( L_2 \) inner product \( \langle x,y \rangle \) is the volume integral \( \int_V x^T y \, dV \) (LeBras and Clayton, 1988). By writing out the integrals for both inner products, one can show that \( \mathbf{F}^T \mathbf{F} \) is a kinematic migration operator; in particular, if \( \mathbf{F} \) is a 2.5-D Kirchhoff modeling operator then \( \mathbf{F}^T \mathbf{d} \) is a poorly-scaled 2.5-D Kirchhoff migration of the recorded data \( \mathbf{d} \), which uses dynamic ray tracing to compute amplitudes and traveltimes.

The most difficult part of the least-squares inversion of equation (6) is to compute \( \mathbf{F}^T \mathbf{F} \). \( \mathbf{F}^T \mathbf{F} \) and \( \mathbf{F} \) are both known, and their product is a migration operator (taking seismic data into a reflectivity model) applied to a modeling operator (taking a reflectivity model into seismic data). This cascade must be realized as a single operator if any efficiency is to be realized. One can approximate \( \mathbf{F}^T \mathbf{F} \) by recognizing that applying a migration operator to the result of a modeling experiment yields, approximately, a scaled, bandlimited identity operator. That is, \( \mathbf{F}^T \mathbf{F} \) applied to a reflectivity spike at the image point \((x, z)\) will produce an image heavily concentrated around \((x, z)\). Because the number of source-receiver pairs illuminating the subsurface changes from one image point to another, \( \mathbf{F}^T \mathbf{F} \) applied to reflectivity spikes placed at different locations within the image space will produce differently scaled bandlimited “spikes.” Thus the diagonal elements of \( \mathbf{F}^T \mathbf{F} \) change from point to point and, by linearity, \( \mathbf{F}^T \mathbf{F} \) applied to a reflectivity model \( \mathbf{m} \) will produce a slightly blurred and unevenly scaled picture of \( \mathbf{m} \). (In fact, the support of \( \mathbf{F}^T \mathbf{F} \) is a narrow band about the main diagonal, but I ignore the contributions from off the main diagonal). Its approximately diagonal nature makes \( \mathbf{F}^T \mathbf{F} \) easy to apply, as \( (\mathbf{F}^T \mathbf{F})^{-1} \) is approximately the diagonal operator whose diagonal elements are given by the reciprocals of the diagonal elements of \( \mathbf{F}^T \mathbf{F} \). These diagonal elements are obtained by applying \( \mathbf{F}^T \mathbf{F} \) separately to reflectivity spikes placed at each image point. This operation accounts for the limited aperture that “sees” each of the image points, and costs slightly more than a prestack Kirchhoff migration (LeBras and Clayton, 1988).

Thus, the linear least-squares migration/inversion is, first, a poorly-scaled migration \( \mathbf{F} \mathbf{d} \) with, next, \( (\mathbf{F}^T \mathbf{F})^{-1} \) approximately correcting the amplitudes by performing an appropriate spatially varying convolution at each image point. The amplitude correction accounts for all, and only, those source-receiver pairs that actually “see” the image point. This improves on the Delft and CWP formulas, whose integrals assume that receivers occupy all surface locations, not just the recording spread, for a given shot. In the Delft and CWP formulas, the surface locations unoccupied by receivers are assumed to contribute traces to the wavefield to be migrated; the replacement of correct data from these locations by zeros introduces migration artifacts and distorts the migrated amplitudes. The linear inversion can be used to begin a sequence of iterations designed to reduce the norm of the residuals \( \mathbf{d} - \mathbf{Fm} \), where \( \mathbf{m} \) is the inverted model at the \( n \)th iteration step. There are several schemes for the iterative solution of linear problems, for example gradient and conjugate gradient methods. To accelerate convergence of the iterations, Sevink and Herman (1994) have suggested using schemes, such as successive overrelaxation, that can use the CWP migration/inversion operator \( \mathbf{F}^{-1} \), rather than \( \mathbf{F}^T \mathbf{F}^{-1} \) as a preconditioner. For these schemes, the first iteration is a CWP migration, with distortions caused by the finite recording aperture removed in later iterations.

LeBras and Clayton (1988) have obtained encouraging numerical results using iterative least-squares to estimate velocity and density model perturbations, and Lambare et al. (1992) have applied the technique to obtain plausible estimates for velocity perturbations along a North Sea line.

**SUMMARY OF THE METHODS, AND DISCUSSION**

The Delft approach assumes that scalar \((P\) and \(S\)) components of the wavefield have been isolated from each other, and that multiples have been removed somehow. It then sets up equation (4), where the observed wavefield is the result of three sequential wavefield operations: downward propagation from the source location(s) to a reflector depth, reflection, and upward propagation from the reflector depth to the recording surface. These operations are expressed simply, and they are
distinct from one another. In processing either the $P$ or the $S$
wavefield component, the generalization of equation (1) is in-
verted exactly (except for stabilizing the method’s handling of
both the evanescent modes and the finite approximations to the
spatial convolutions) to yield a reflectivity vector at each im-
age location. Because the basic molecular unit of seismic data
in the Delft approach is a monochromatic (single frequency)
shot record, the migration/inversion involves a wavefield ex-
trapolation in the frequency-space ($F - X$) domain. This ex-
trapolation method allows the velocity to vary from point to
point; the wavefield is extrapolated from a point $(x, z)$ on one
depth level to a small aperture on the next depth level us-
ing the velocity at $(x, z)$. The nature of $F - X$ extrapolation
operators requires an underlying wave equation in either two
or three dimensions, as 2.5-D inversion is not realizable. In
two dimensions, the Delft wavefield extrapolation operators
are Hankel functions, which are the Green’s functions for the
2-D wave equation. These are accurate for structural imag-
ing, although the accuracy of their amplitudes becomes ques-
tionable when large velocity contrasts force the use of ad hoc
transmission factors. Limited to wave-equation domains, such
as common-shot or generalized plane-wave records (Rietveld
and Berkhout, 1994), the method cannot be applied to com-
mon offset records. When applied to common-shot records, it
can suffer from migration artifacts caused by data truncations
at the edges of the finite-length recording spread. Because the
spread typically does not span the entire line, these artifacts
will be distributed uniformly across the migrated section, to
the detriment of the analysis of amplitudes on migrated common-
reflection-point (CRP) gathers. When applied to generalized
plane-wave records, it will suffer from similar artifacts; but be-
cause the plane-wave records span the entire line, the artifacts
will be concentrated at the edges of the section, and will not
interfere with the analysis of migrated amplitudes on events in
the middle of the section. (Generalized plane-wave records ob-
tained by summing physical shot records can themselves suffer
from finite-aperture artifacts; let us assume here that this prob-
lem has been solved.) Finally, the Delft migration/inversion
method is theoretically simple, and relatively easy to program.

CWP migration/inversion fits between the Delft and the
least-squares approaches. In contrast to the Delft approach,
the CWP expression for the recorded wavefield appears as a
single complicated linear integral equation; however, this com-
licated expression bears a close relationship to the Delft
expression $P = W_{ap} R W_{disp} S$. In contrast to the simple deriva-
tion of the least-squares migration/inversion formula, CWP
migration/inversion is an analytical tour de force. However,
the two migration/inversion formulas are related very closely.
The CWP inversion of the complicated wavefield expression
involves a great deal of analysis, including the use of high-
frequency asymptotics in several places. The resulting migra-
tion/inversion formula is familiar to us as a Kirchhoff
migration with weights appropriate for any chosen processing domain
(common-shot, common-offset, etc.). Familiar as the basic mi-
gration formula is, it is still mysterious in detail because of
subtleties in the derivation. Kirchhoff migration, which uses
ray-theoretic rather than exact extrapolation operators, is usu-
ally less accurate than the Delft $F - X$ migration, and produces
a specular reflectivity value, not a vector, at each image point.
Nevertheless, it is more efficient than $F - X$ migration, and it is
available in a 2.5-D form. Also, it can be modified to produce
an estimate of incidence angle at a reflector, and allows the in-
corporation of transmission losses into the ray-amplitude cal-
culations more readily than the Delft approach. Like the Delft
method, it suffers from limited aperture artifacts, which can be
especially damaging when applied to shallow reflectors, or
when applied to common-shot records. Because of intricacies
of ray tracing and aperture calculations, it is not easy to pro-
gram.

The least-squares method interprets the CWP integral equa-
tion for the reflectivity $R$ as a linear equation, to be inverted for
$R$ in the least-squares sense. The resulting migration/inversion
formula agrees with the CWP formula in the unrealizable case
where complete information (infinite recording aperture) is
available; where complete information is unavailable, it cor-
rects the problems caused by incomplete recording geometry.
This is the chief advantage of the least-squares approach over
both of the other methods. This advantage should be most ap-
parent where the effects of limited aperture are the greatest,
for example in processing common-shot records or common-
receiver records near the edges of the section, or vertical
seismic profile (VSP) records. Least-squares inversion leads
to a Kirchhoff migration weighted differently from the CWP
Kirchhoff migration. The implementation of this approach is
at least slightly more complicated and slightly less efficient than
the CWP migration/inversion. Some implementations involve
iterative solutions of the linear equation, which can make them
time-consuming.

Having discussed the major similarities and differences
among the methods, let us now summarize the practical
limitations pointed out in the Introduction. They are listed
in the order of importance that I perceive; thus, they reflect
my experience and biases. First, the lack of a 2.5-D formula is
a disadvantage of the Delft method. Second, the effects of
limited aperture can be serious. The least-squares approach
handles this problem better than the other two methods do
and, in my opinion, is the preferred true-amplitude migra-
tion because of this. However, the shortcomings of the other
two methods can be minimized by processing reflection data
in an appropriate domain (plane-wave domain for the Delft
method, and common-offset domain for the CWP method),
thereby pushing the artifacts away from the imaging objec-
tives in the center of the section and minimizing the effects of
aperture limitations. Third, the inability to deal with caustics
casted by velocity variations seriously compromises the credi-
bility of the CWP and least-squares methods. In some instances,
moderate velocity variations can cause caustics in the wave-
fields; these instances can invalidate migration/inversion re-
sults without the user knowing. Fourth, least-squares inversion
yields, in principle, a more complete model of elastic parame-
ters than either the Delft or the CWP approach can provide.
The least-squares computed elastic parameter perturbation
function can simply be combined with the background model to pro-
duce a complete map of elastic parameters, whereas the Delft
can and CWP methods provide, at most, a way to estimate pa-
terms jumps at isolated reflector locations. Of course, the
least-squares map of parameters is only as reliable as the back-
ground model, which is not easily estimated from the seis-
mic data alone. Fifth, none of the methods deals adequately
with either anisotropic effects or transmission losses. How-
ever, where these effects are important, the very application of
lossless, elastic, true-amplitude migration is problematic. Sixth,
the Delft approach has the theoretical advantage of producing a reflectivity operator, while the other methods produce reflection coefficients. Given all the uncertainties present in seismic data, and the fundamental limitations to true-amplitude migration listed in the Introduction, it is unlikely that this advantage can be put to practical use.

To summarize, migration without consideration of amplitudes attempts to provide the transpose of a forward modeling operator [such as \( \mathbf{F} \) in equation (6)], while true-amplitude migration attempts to provide the inverse of the operator, producing reflection coefficients or medium parameters at actual earth locations. Despite the many shortcomings in the methods discussed here, true-amplitude migration has greater potential to exploit the amplitude information present in seismic data than does migration that does not account for the amplitudes of the wavefields. A fortiori, AVO analysis of true-amplitude migrated records should be considered preferable to standard AVO analysis of unmigrated records, which neither migrates reflection events to their correct spatial locations nor undoes the amplitude and phase distortions caused by wave propagation within the earth.

Although several workers have tested these methods on data sets (mostly synthetic), it remains to test them systematically on a series of synthetic and field data examples, ranging from simple to realistic. These tests will include the effects of caustics in the wavefields as well as small-scale variations in velocity and small-scale components of reflector roughness. Only after such calibration will the power and limitations of true-amplitude migration be well understood.

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