Reverse-time migration: amplitude and implementation issues
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Summary
We formulate reverse-time migration (RTM) based on the theory of true-amplitude migration, and we give formulations for true-amplitude RTM angle-domain common-image gathers. Then we address some implementation issues for RTM. Specifically, we compare RTM’s efficiency using different orders of finite differencing along the time direction. Finally, we propose “harmonic-source migration”, a phase-encoding technique that allows increased efficiency in a delayed-shot implementation of RTM.

Introduction
Prestack depth migrations based on one-way wave equations (Claerbout, 1971) have been widely used in 3-D seismic processing because they are efficient and are capable of imaging complex structures. Recent research and development have calibrated one-way wave equations to produce friendly migrated amplitudes (Zhang et al., 2005b; Zhang et al., 2007) and to image steep dips using turning-waves (Zhang et al. 2006b). An alternative, migrating data directly with the two-way wave equation, is also attractive. Such a method has existed for a long time and is called reverse-time migration (RTM) (Whitmore, 1983; Baysal et al., 1983). Unlike one-way wave equation migration, reverse-time migration does not need to deal with the theory of singular pseudo-differential operators. A straightforward implementation of RTM can handle complex velocities to produce all kinds of acoustic waves (reflections, refractions, diffractions, multiples, evanescent waves, etc.), can carry correct propagation amplitude, and imposes no dip limitations on the image. When the migration artifacts produced by RTM are suppressed by some post-migration processing, the only remaining problem with this method is its efficiency. In this abstract, we first formulate RTM based on inversion theory and then we address some implementation issues. Then we compare the relative efficiency of using different orders of finite differencing along the time direction. At the end, we propose harmonic-source migration as a way to improve the efficiency of delayed-shot RTM.

True-amplitude reverse-time prestack depth migration
We first formulate RTM based on the theory of true-amplitude migration. To migrate a shot record \(Q(x, y, z; t)\) with the shot at \((x, y, z, t) = 0\) and receivers at \((x, y, z) = 0\), we have to compute the wavefields originating at the source location and observed at the receiver locations. Because the source wavefield expands as time increases and the recorded wavefield is computed backward in time, we denote them by \(p_f\) and \(p_b\) in the following two-way wave equations:

\[
\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) p_f(x; t) = \delta(\mathbf{x} - \mathbf{x}_s) f(t), \tag{1}
\]

and

\[
\left\{ \begin{aligned}
\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) p_b(x; t) &= 0, \\
p_b(x, y, z = 0; t) &= Q(x, y, x_s, y_s, t),
\end{aligned} \right. \tag{2}
\]

where \(c = c(x, y, z)\) is the velocity, \(f(t)\) is the source signature, and \(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\) is the Laplacian operator. To obtain common-shot image with correct migration amplitude, we need to apply the “deconvolution” imaging condition (Zhang et al., 2005b)

\[
R(\mathbf{x}) = \int p_b(\mathbf{x}; t) p_f^{-1}(\mathbf{x}; t) dt, \tag{3}
\]

where \(p_f^{-1}(\mathbf{x}; t)\) is defined as the inverse of the wavefield \(p_f(\mathbf{x}; t)\), i.e.

\[
\Psi_f \ast p_f^{-1}(\mathbf{x}; t) = \delta(\mathbf{x}; t). \tag{4}
\]

The imaging condition (3) is simple to apply in the frequency domain for one-way wave equation migration. However, it is difficult to implement in the time domain for RTM. In practice, the “cross-correlation” imaging condition

\[
R(\mathbf{x}) = \int p_b(\mathbf{x}; t) p_f(\mathbf{x}; t) dt \tag{5}
\]

is often preferable for reasons of stability. Although this is not appear to be consistent with true-amplitude migration, Zhang et al. (2007) proved that the imaging condition (5) is a proper choice to obtain true-amplitude angle gathers from wave equation based migration. For this to occur, equation (1) need to be modified accordingly as

\[
\left\{ \begin{aligned}
\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) p_f(x; t) &= 0, \\
p_f(x, y, z = 0; t) &= \delta(\mathbf{x} - \mathbf{x}_s) \int f(t') dt'.
\end{aligned} \right. \tag{6}
\]

This equation is different from the conventional wave equation (1) for the forward wavefield, because the source at the surface is treated as a boundary condition instead of a right-hand-side forcing term of the equation.

In summary, we propose the following algorithm to output true-amplitude angle-domain common-image gathers from RTM:
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1. Compute forward and backward wavefields $p_k$ and $p_{k_b}$ by solving the two-way wave equations (6) and (2);

2. Apply the cross-correlation imaging condition (5) during the migration;

3. Use an existing method, e.g. Sava and Fomel (2003), to output angle-domain common-image gathers.

The migration output then provides angle dependent reflectivity in the sense of high frequency approximation.

Explicit schemes for solving two-way wave equations

To perform prestack RTM, we first temporally extrapolate the forward wavefield $p_k$ at all subsurface locations by solving the two-way wave equation (6) in time; we store this four dimensional wavefield. Then we propagate the recorded seismic data $Q(x, y, z, t)$ backwards in time and generate the backward wavefield $p_{k_b}$ at all the subsurface locations by solving equation (2). During this process, we apply the imaging condition (5) to the two wavefields to produce the image.

The most expensive part of RTM is solving the two-way wave equations, for example, equation (2):

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) p(x; t) = 0. \quad (7)$$

Using the pseudo-spectral method, we express equation (7) as

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + F^{-1}(k_x^2 + k_y^2 + k_z^2) F \right) p(x; t) = 0, \quad (8)$$

where $k_x, k_y$ and $k_z$ are wavenumbers in $x$, $y$ and $z$, respectively. The operators $F$ and $F^{-1}$ are forward and inverse Fourier transforms. Equation (8) is usually solved by an explicit finite-difference scheme in time. In this case, the efficiency of the computation is determined by the stability condition and the numerical dispersion relation. If we write

$$P^n \Phi = p(x; n\Delta t), \quad (9)$$

the second-order explicit finite-difference scheme for equation (8) is

$$P^{n+1} - 2P^n + P^{n-1} \frac{c^2 \Delta t^2}{\Delta x^2 + \Delta y^2 + \Delta z^2} = -F^{-1}(k_x^2 + k_y^2 + k_z^2) F P^n. \quad (10)$$

This equation needs to satisfy the stability condition

$$\Delta t \leq \frac{2}{c_{\text{max}} \sqrt{\frac{\Delta x^2}{\Delta t^2} + \frac{\Delta y^2}{\Delta t^2} + \frac{\Delta z^2}{\Delta t^2}}}. \quad (11)$$

and gives the numerical dispersion error in the frequency domain

$$1 - \frac{4 \sin^2 \left( \frac{\omega \Delta t}{2} \right)}{\omega^2} = 1 - \left( \frac{N_p \sin \frac{\pi}{N_p}}{\pi} \right)^2, \quad (12)$$

where $N_p$ is the number of samples per wave length.

Better accuracy can be achieved through the fourth-order finite-difference scheme (Etgen, 1986):

$$\begin{align*}
\frac{P^{n+1} - 2P^n + P^{n-1}}{c^2 \Delta t^2} &= -F^{-1}(k_x^2 + k_y^2 + k_z^2) F P^n \\
&\quad + \frac{\Delta t^2}{12c^4} F^{-1}(k_x^2 + k_y^2 + k_z^2) F P^n.
\end{align*} \quad (13)$$

Equation (13) needs to satisfy the corresponding stability condition

$$\Delta t \leq \frac{2\sqrt{3}}{c_{\text{max}} \sqrt{\frac{\Delta x^2}{\Delta t^2} + \frac{\Delta y^2}{\Delta t^2} + \frac{\Delta z^2}{\Delta t^2}}}, \quad (14)$$

with its dispersion error

$$1 - \frac{6 - 6 \left( \frac{4 \sin^2 \left( \frac{\omega \Delta t}{2} \right)}{\omega^2} \right)}{2} = 1 - \frac{3}{2} \left( \frac{N_p}{\pi} \right)^2 \left( 1 - \frac{4 \sin^2 \frac{\pi}{N_p}}{N_p} \right). \quad (15)$$

![Figure 1: Dispersion errors for the second-order (blue) and the fourth-order (purple) finite-difference scheme.](image)

In Figure 1, we plot the dispersion error for the second-order and the fourth-order schemes. A 1% error threshold requires 5 points to sample a full wavelet for the fourth-order finite-difference scheme, while for the same threshold, 18 points are requires for the second-order scheme to achieve the same accuracy. Given the highest...
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frequency $f_m$ in the seismic data, we have the following criteria to determine the time step size:

$$\Delta t = \min\left(\frac{2}{c_{\text{max}}} \sqrt{\frac{\pi^2}{\Delta x^2} + \frac{\pi^2}{\Delta y^2} + \frac{\pi^2}{\Delta z^2}}, \frac{1}{18 f_m}\right)$$

(16)

for the second-order scheme, and

$$\Delta t = \min\left(\frac{2\sqrt{3}}{c_{\text{max}}} \sqrt{\frac{\pi^2}{\Delta x^2} + \frac{\pi^2}{\Delta y^2} + \frac{\pi^2}{\Delta z^2}}, \frac{1}{5 f_m}\right)$$

(17)

for the fourth-order scheme. Comparing equations (16) and (17), and analyzing the second-order and the fourth-order finite-difference schemes, we conclude that the fourth-order scheme is superior to the second-order scheme in efficiency to obtain the same migration accuracy.

Delayed-shot, plane-wave, and harmonic-source RTM

For common-shot migration, the cost equals the cost of migrating a single shot times the number of shot migrations. Various approaches have been proposed to reduce the number of shots, thus reducing the project cycle time and cost. Combining shots by line-source synthesis in the inline direction (delayed-shot migration) or in both inline and crossline directions (plane-wave migration) produces satisfactory results if enough $p$-values are used (Whitmore 1995; Duquet et al. 2001; Liu, 2002; Zhang et al. 2005a; Etgen, 2005). One may consider applying similar techniques to RTM to improve its efficiency. Taking delayed-shot migration for example, we need to apply a $\tau p$ transform to the input data to synthesize the line source response, i.e.

$$q(x, y; p, t) = \int Q(x, y; x_1, y_1; t - p_1) dx_1.$$  

(18)

For migrations performed in the frequency domain, the time delays in delayed-shot migration can be implemented as phase shifts, avoiding the need for time padding of the input traces. Since we perform RTM in the time domain, delayed-shot RTM requires a long time padding for long sail lines and large values of $p$s; this can slow down the process considerably. To avoid this problem, we propose a phase-encoding algorithm, which we call harmonic-source migration. This migration is equivalent to delayed-shot migration but does not suffer from the long time padding problem. We have applied this migration to both one-way wave equation migration and RTM. For harmonic-source migration, the phase encoding function is in the time domain

$$h(x; k) = \cos k x_1 \hat{t}(t) - \sin k x_1 \frac{2}{l},$$

(19)

so there is negligible time padding to apply during the $x \rightarrow k$ spatial transform. The number of $k$’s can be determined by a similar analysis to the number of $p$’s in delayed-shot migration (Zhang et al., 2005a; Zhang et al. 2006a). This leads to a speedup ratio of harmonic-source migration versus common-shot migration of

$$R = \frac{L_s}{h_{\text{max}}},$$

(20)

where $L_s$ is the size of the aperture of common-shot migration in the $x$-direction and $h_{\text{max}}$ is the maximum offset in the $x$-direction. For a typical production project, if $L_s = 18km$ and $h_{\text{max}} = 8km$, the speedup ratio $R = 2.25$.

Numerical examples

To show how true amplitude angle domain RTM works, we apply it to a 2-D horizontal reflector model in a medium with velocity $c = (2000 + 0.3 \cdot z)m/s$. Figure 2 shows a single shot record over five horizontal reflectors. The shot is in the center of the section and the receivers cover the surface in an aperture of 15000m on each side. The amplitude variation across travel time and lateral distance is due only to geometrical spreading loss. We migrated the shot records using the common-shot RTM algorithm (2) and (6) with the imaging condition (5). At an image location, we stack all the migrated common image shot gathers to generate the subsurface offset gathers, and then use the method proposed by Sava and Fomel (2003) to convert them to the subsurface reflection angle gathers shown in Figure 3. The normalized peak amplitudes along the reflectors in the angle domain are shown in Figure 4. It is clear that the amplitudes in the angle domain recover the reflectivity accurately over a large angular range, aside from edge effects.

In the second example, we apply RTM to the 2004 BP 2D data set (Billette and Brandsberg-Dahl, 2005) and compare common-shot migration (left) versus harmonic-source migration (right) in Figure 5. This is a high quality dataset generated by finite-difference modeling with shot spacing 50m, receiver spacing 12.5m and 15000m maximum offset. For such a data set, the two RTMs give almost identical images. Both give good delineation of the salt boundaries, especially the steeply dipping salt flanks and the overturned salt edges, which require high angle propagation or turning waves to image clearly.

Conclusions

We have analyzed the amplitude behavior and efficiency of reverse-time migration. We have shown that modifying the initial-value problem for the one-way wave equation for the
source wavefield into a boundary-value problem for the two-way wave equation for the same wavefield, plus implementing an appropriate imaging condition, yields a true-amplitude version of RTM. We have also shown that using fourth-order finite-differences in time to achieve a given accuracy is more efficient than using second-order finite differences. Finally, we have introduced a “harmonic-source” phase encoding method to allow a relatively efficient delayed-shot or plane-wave RTM.

Taken together, these incremental advances yield a powerful true-amplitude migration method that uses the complete two-way acoustic wave equation.

We have illustrated the amplitude behavior of RTM on a flat-layered synthetic model data set, and we have illustrated its structural imaging capabilities of harmonic-source RTM on complex structural data sets.

**Figure 3**: A migrated angle domain common-image gather.

**Figure 4**: Normalized peak amplitude versus reflection angle curves along the migrated reflectors in Figure 2.

**Figure 5**: A comparison of common-shot (left) and harmonic-source (right) RTM on 2004 BP 2D data set.
References


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