One-step extrapolation method for reverse time migration

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**ABSTRACT**

We have proposed a new method, a one-step extrapolation algorithm, to solve the acoustic wave equation. By introducing a square-root operator, the two-way wave equation can be formulated as a first-order partial differential equation in time, which is similar to the one-way wave equation. To solve the new wave equation, we used a stable explicit extrapolation method in the time direction and handled lateral velocity variations in the space and wavenumber domains. Unlike the conventional explicit finite-difference schemes, the new method does not suffer from numerical instability or numerical dispersion problems. It can be used to design cost-effective and high-quality reverse time migration or modeling code.

**INTRODUCTION**

Reverse time migration (RTM) based on directly solving the acoustic wave equation (Whitmore, 1983; Baysal et al., 1983; McMechan, 1983) provides a natural way to deal with large lateral velocity variations and imposes no dip limitations on the images. Recently, it has attracted considerable attention and is considered to be a method of choice for imaging complex structures. In practice, reverse time migration is implemented by solving the acoustic wave equation with different finite-difference (FD) schemes that fall into two main categories: implicit and explicit FD.

A conventional implicit FD scheme requires solving a linear system of equations equal in size to the product of the dimensions of the migration volume, which is much larger than the memory and computational cost needed by explicit schemes. Therefore, explicit FD schemes are used almost exclusively in 3D reverse time migrations. Although explicit FD schemes are easy to solve, theoretical analysis shows that they are stable only when a limit is imposed on the size of the marching time step. However, both FD methods suffer from numerical dispersion problems. To overcome these problems, either high-order temporal and spatial FD schemes are used, or the spatial grid size and time step size are reduced. In either case, the computational cost increases.

Here, we propose a new way of solving the acoustic wave equation, which we call the one-step extrapolation (OSE) method. This method differs from conventional FD methods in that it does not suffer from numerical instability and dispersion problems. The new method is based on reformulating the two-way wave equation by introducing a complex (analytic) pressure wavefield. The solution of the new equation can be expressed symbolically and then computed by a stable explicit extrapolator derived from an optimized separable approximation (OSA). Numerical examples show that OSE-based reverse time migration has the capability to image steeply dipping reflectors and complex structures.

**THEORY**

The most expensive part of RTM is solving the acoustic wave equation for the forward and reverse wavefields:

$$\left(\frac{\partial^2}{\partial t^2} - V^2 \Delta\right) p(\mathbf{x}; t) = 0,$$

where $p$ is the pressure wavefield, $V$ is velocity, and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian operator.

For constant velocity, after applying the spatial Fourier transform $\tilde{p}(\mathbf{k}; t) = Fp(\mathbf{x}; t)$, the acoustic wave equation 1 can be rewritten as

$$\left(\frac{\partial^2}{\partial t^2} + \varphi^2\right)\tilde{p}(\mathbf{k}; t) = 0,$$

where $\varphi = V\sqrt{k_x^2 + k_y^2 + k_z^2}$. Now we define a complex pressure wavefield $\mathbf{P}$ as

$$\mathbf{P}(\mathbf{x}; t) = p(\mathbf{x}; t) + iq(\mathbf{x}; t),$$

where $q$ is the Hilbert transform of $p$, so in the frequency domain, $\tilde{q}(\mathbf{k}; \omega) = i \text{sgn}(\omega)\tilde{p}(\mathbf{k}; \omega)$. From the dispersion relation $\omega^2/V^2 = k_x^2 + k_y^2 + k_z^2$ of the acoustic wave equation, $\tilde{q}(\mathbf{k}; \omega)$ can be expressed further as.
\[ \hat{q}(k; \omega) = i \text{sgn}(\omega) \hat{p}(k; \omega) = \frac{i \omega}{\omega} \hat{p}(k; \omega) = \frac{i \omega}{\sqrt{k_x^2 + k_y^2 + k_z^2}} \hat{p}(k; \omega), \] (4)

or
\[ \hat{q}(k; t) = \frac{1}{\varphi} \frac{\partial \hat{p}(k; t)}{\partial t}. \] (5)

Therefore, equation 2 is equivalent to the following equation system,
\[ \frac{\partial}{\partial t} \left( \frac{\hat{p}}{i \varphi} \right) = \begin{pmatrix} 0 & -i \varphi \\ -i \varphi & 0 \end{pmatrix} \begin{pmatrix} \hat{p} \\ i \varphi \hat{p} \end{pmatrix}, \] (6)

or simply,
\[ \frac{\partial \hat{P}}{\partial t} = \frac{\partial \hat{P} + i \varphi}{\partial t} = -i \varphi (\hat{P} + i \varphi) = -i \varphi \hat{P}. \] (7)

For general media, similar to equation 7, the complex pressure wavefield \( \hat{P} \) satisfies the following first-order partial differential equation in the time direction:
\[ \left( \frac{\partial}{\partial t} + i \Phi \right) P(x; t) = 0. \] (8)

Here, \( \Phi = V \sqrt{-\Delta} \),

or by its symbol,
\[ \varphi = V(x, y, z) \sqrt{k_x^2 + k_y^2 + k_z^2}. \] (10)

We can verify that in a general medium, equation 8 is equivalent to the two-way wave equation 1 in the asymptotic sense, which means that OSE gives the same travelt ime and leading order amplitude as the conventional acoustic wave propagation does (Bleistein et al., 2008). Gazdag (1981) proposed a wave equation similar to equation 8,
\[ \left( \frac{\partial}{\partial t} + i \frac{k_z}{k_z} \right) P(x; t) = 0. \] (11)

However, because the real pressure wavefield is used in equation 11, the propagation angle is limited to 90°, just as with the conventional one-way wave equation (Claerbout, 1971; Gazdag, 1978).

### NUMERICAL IMPLEMENTATION

If the velocity is constant, the solution of equation 8 can be expressed easily as
\[ P(x; t + \Delta t) = (F^{-1} e^{-i \Delta \Phi V / \sqrt{k_x^2 + k_y^2 + k_z^2}} F) P(x; t). \] (12)

As discussed in the previous section, equation 12 is an analytic solution of acoustic wave equation 2 for arbitrarily big time step size \( \Delta t \). For variable velocity, the solution for equation 8 can be written symbolically as
\[ P(x; t + \Delta t) = e^{-i \Delta \Phi} P(x; t). \] (13)

Recall that the solution of the one-way wave equation (Claerbout, 1971; Gazdag, 1978) can be expressed as
\[ D(z + \Delta z) = e^{-i \Delta \lambda} D(z), \] (14)

where \( \lambda \) is the square-root operator with the symbol,
\[ \lambda = \sqrt{\frac{\omega^2}{V^2(x, y, z)} - k_x^2 - k_y^2}. \] (15)

Because of the similarity between the two solutions 13 and 14, any method for computing the one-way operator \( e^{-i \Delta \lambda} \), such as phase shift plus interpolation (PSPI) (Gazdag and Sguazzero, 1984), non-stationary phase shift (NSPS) (Margrave and Ferguson, 1999), and explicit extrapolation (Hale, 1991a, 1991b), can be applied with appropriate changes to solve the two-way wave equation 13. The difference is that for the one-way wave equation, the evolution direction is depth \( z \), whereas for the two-way wave equation it is time \( t \). The symbol \( \lambda \) for the one-way wave equation is singular near \( \omega^2 / V^2 = \frac{k_x^2 + k_y^2}{k_z^2} \) (where the propagation wave becomes evanescent), whereas the operator \( \varphi \) in equation 13 has no singularity except for a single point \( k_x^2 + k_y^2 + k_z^2 = 0 \) and hence is better behaved. This fact allows a computational convenience when solving equation 13 numerically.

Here, we introduce a method based on an optimized separable approximation (OSA) proposed by Song (2001). Given the time step \( \Delta t \), and the velocity \( V \) and the wavenumber variation ranges, \( V \in [V_{\min}, V_{\max}] \) and \( k \in [k_{\min}, k_{\max}] \), the goal is to approximate the two-way propagator in a separable form:
\[ e^{-i V \Delta t} \approx \sum_{n=1}^{N} a_n(V) b_n(k), \] (16)

where \( a_n(V) \) and \( b_n(k) \) are complex functions of velocity and wave-number, respectively. They can be determined by computing the left and right eigenfunctions of the two-dimensional function
\[ A(V, k) = \exp(-i V k \Delta t). \] (17)

For example, the first approximation pair \( a_1(V) \) and \( b_1(k) \) are given by
\[ a_1(V) = \lambda_1 \phi_1(V), \] (18)

and
\[ b_1(k) = \psi_1(k), \] (19)

where \( \lambda_1 \) is the first eigenvalue of the operator 17, and \( \phi_1(V) \) and \( \psi_1(k) \) are its left and right eigenfunctions. For details, we refer to Song (2001) and Chen and Liu (2004). Chen and Liu (2006) applied this method to solve the one-way wave equation. Because the two-way wave operator 13 is well behaved, OSA exponentially converges to it, as proved by Song (2001). This seems an ideal way to provide a fast solver for the wave equations.

After we obtain the OSA in equation 16, the wave propagation 13 can be performed by OSE in both space and wavenumber domains, i.e.,
\[ P(x; t + \Delta t) = \left( \sum_{n=1}^{N} a_n(V) F^{-1} b_n(k) \right) F P(x; t). \] (20)

Figure 1 is a flowchart of the algorithm. For every marching time step, the algorithm requires one fast Fourier transform (FFT) and \( N \) inverse fast Fourier transforms (IFFTs), which are the most computationally intensive part in OSA. As in the pseudospectrum method, performing the operations in wavenumber domain means that OSE will not suffer from numerical dispersion. In addition, OSE is guaranteed to be stable for any given time step size \( \Delta t \) because we design the algorithm by approximating the analytic solution (equation 16) for all possible velocity and wavenumber variation ranges.
NUMERICAL EXAMPLES

In the first example, we use OSE methods to migrate a 3D impulse response in a medium with velocity \( V = (2000 + 0.5z) \) m/s, where \( 0 \leq z \leq 5000 \) m, so the velocity varies from 2000 m/s to 4500 m/s, and the maximum wavenumber is \( \sqrt{3}/40 \pi \). The grid sizes of propagation are \( \Delta x = \Delta y = \Delta z = 40 \) m. In OSE reverse time migrations, the marching time step \( \Delta t \) can be chosen arbitrarily. Because the maximum frequency in the input data is 25 Hz, we set \( \Delta t = 20 \) ms to avoid sampling aliasing. With this parameter setup, the OSA uses only six terms to converge to the two-way operator with a maximum error drops almost exponentially with the number of approximation terms.

\[
P_{n+1} - 2P_n + P_{n-1} + \left( V^2 \Delta^2 F^{-1} |k|^2 F \right) P^n = 0, \quad (21)
\]

\[
P_{n+1} - 2P_n + P_{n-1} + \left( I - \frac{V^2 \Delta^2 F^{-1} |k|^2 F}{12} \right) \times \left( V^2 \Delta^2 F^{-1} |k|^2 F \right) P^n = 0, \quad (22)
\]

The time steps should be smaller than 2.22 ms and 5.66 ms (Etgen, 1986; Zhang et al., 2007a), respectively, to avoid numerical instability and dispersion.

Figure 1. The flowchart of using OSA algorithm to solve one-step exploration equation 13.

To compare the computational cost of OSE algorithm 20 with those of FD schemes 21 and 22, we see that for all three algorithms, the most computationally intensive parts are the FFT and IFFT. Therefore we need only to count the total number of FFT and IFFT operations in each algorithm for a given maximal extrapolation time \( T \) ms, which will be \( T/7 \) for OSE, \( T/2.22 \) for the second-order FD, and \( 4 \times T/5.66 \) for the fourth-order FD, respectively. Therefore, for this example, the OSE method is about 2.6 times faster than the second-order FD and two times faster than the fourth-order FD method.

Figure 3 shows the migrated impulse response at the central inline. It is clear that the image has all possible dips, even beyond 90°, which means that OSE gives a complete solution to the two-way wave equation. Figure 4 shows a migrated depth slice. The slice is perfectly circular and has no obvious numerical dispersion.

Figure 3. An inline image of the impulse response test. Velocity \( V = 2000 + 0.3z \).

Figure 4. A depth slice at \( z = 1 \) km of the impulse response (Figure 3).
In the second example, we apply OSE reverse time migration to the 2004 BP 2D data set (Billette and Brandsberg-Dahl, 2005). This is a high-quality data set generated by FD modeling with shot spacing of 50 m, receiver spacing of 12.5 m, and 15,000-m maximum offset. In the migration, the highest frequency is 30 Hz, so the time step $\Delta t = 16.6$ ms was used. For such a data set, the OSE method handles complex velocity fairly well and gives good delineation at the salt boundaries (Figure 5), especially at the steeply dipping salt flanks and the overturned salt edges, which require high angle propagation or turning waves to obtain a clear image.

**DISCUSSION**

The OSE method was presented first in a paper by Zhang et al. (2007b). Bleistein et al. (2008) later analyzed the operator $\Phi$, and proved that equation 8 above is asymptotically equivalent to the acoustic wave equation 1. A proper algorithm to compute the pseudodifferential operator $\Phi$ is needed to handle rapidly varying velocity. The simple OSA algorithm described here does not handle sharp velocity boundaries well. It shows oscillations when energy is propagated through areas with high velocity contrast. Soubaras and Zhang (2008) proposed a two-step extrapolation method to overcome the problem. The two-step extrapolation method does not require a complex wavefield and can handle any velocity contrast. In a private discussion, Paul Fowler showed Yu Zhang that he had developed a similar method independently and applied it to anisotropic wave equation modeling (P. Fowler, personal communication, 2007).

**CONCLUSION**

Reverse time prestack depth migration is a powerful tool to image complex structures, but it is still computationally intensive. The one-step extrapolation method we propose provides a new way of solving two-way wave equations. The OSE method uses an optimized approach to solve the wave equation by approximating an analytic solution. It allows a large extrapolation time step and so does not suffer from numerical instability and dispersion problems normally found in conventional explicit finite-difference algorithms. Therefore, it can be used in designing cost-effective, high-quality modeling and imaging software.

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