**Amplitude calculations for 3-D Gaussian beam migration using complex-valued traveltimes**

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**Summary**

Gaussian beams are often used to represent Green’s functions in three-dimensional Kirchhoff-type true-amplitude migrations because such migrations yield superior images to similar migrations using classical ray-theoretic Green’s functions. Typically, the integrand of a migration formula consists of two Green’s functions, each describing propagation to the image point—one from the source position and the other from the receiver position.

The use of Gaussian beams to represent each of these Green’s functions in 3D introduces two additional double integrals when compared to a Kirchhoff migration using ray-theoretic Green’s functions, thereby adding a significant computational burden. Hill [2001] proposed a method for reducing those four integrals to two, compromising slightly on the full potential quality of the Gaussian beam representations for the sake of more efficient computation. That approach requires a two-dimensional steepest descent analysis for the asymptotic evaluation of a double integral. The method requires evaluation of the complex traveltimes of the Gaussian beams as well as the amplitudes of the integrands at the determined saddle points. In addition, it is necessary to evaluate the determinant of a certain (Hessian) matrix of second derivatives. Hill [2001] did not report on this last part; thus, his proposed migration formula is kinematically correct but lacks correct amplitude behavior. Here we report on the derivation of a formula for that Hessian matrix in terms of dynamic ray tracing quantities.

**Introduction**

True-amplitude Kirchhoff migration requires downward propagation of the source wavefield and downward propagation of the observed data to image points in the subsurface. The downward propagations are accomplished using Green’s identity, operating on data evaluated along one surface to obtain data at a deeper surface via a convolution-type integral of the data with a Green’s function. We also also downward propagate the sources other than the point source we assume in this report as a convolution-type integral. This is Kirchhoff migration [Schneider, 1978]. Each of these propagation processes requires evaluation of Green’s functions at the image point, one from a source, one from a receiver. When Gaussian beam representations of these Green’s functions are used, it is necessary to generate the Gaussian beams themselves in neighborhoods of the image points. Then, to obtain the Gaussian beam representation of each Greens function in 3D, it is necessary to carry out a 2D integration of Gaussian beams over all takeoff angles where rays are in the vicinity of the image point. By contrast, the Greens functions for standard Kirchhoff migration, derived from classical asymptotic ray theory, require only a complex function evaluation with no additional integrations.

Multiplying the Green’s functions together, as required by migration theory, results in the need to evaluate four nested integrals for Gaussian beam migration, as opposed to the multiplication of two complex numbers for classical ray-theoretic Green’s functions. Hill [2001] suggested a method for reducing those four additional integrals to two. Hill’s method first replaces integrals over source and receiver ray parameters with integrals over midpoint and offset ray parameters. He then applies the method of steepest descent for integrals with complex exponents [Bleistein, 1984] to the (innermost) integrals over offset parameters, leaving the (outermost) integrals over midpoint parameters to be computed numerically. (The method of steepest descent applies to single integrals in the complex plane. Here, we need an extension of that method to a double integral. That extension is presented by Bleistein and Gray [2010].) Hill [2001] provides a technique for determining the critical (saddle) points and evaluating the complex traveltime and amplitude in the Kirchhoff integral formula. However, for true-amplitude integrity, the steepest descent approximation of the integral also requires including the determinant of the Hessian matrix of second derivatives of the complex traveltime with respect to the two offset ray parameters as an adjustment factor in the amplitude of the asymptotic approximation. Hill [2001] did not evaluate that determinant; thus his method suffers a slight degradation in amplitude and phase fidelity. That is, its peak amplitude on reflectors cannot be guaranteed to be proportional to a specular reflection coefficient.

The purpose of this presentation is to report on the evaluation of the determinant of that Hessian. It can be written as the sum of Hessian matrices of complex traveltime with respect to the initial transverse source slownesses and with respect to the initial receiver slownesses. We do not have direct knowledge of those Hessians in Cartesian variables. However, we do know how the Hessians in the kinematic variables \( q = (q_1, q_2) \) (ray-centered coordinates) propagate; those can be expressed in terms of the dynamic quantities of ray theory in those variables. Thus, we need to transform the Hessians in the Cartesian slownesses—one for the source variables and one for the receiver variables—into the Hessians in ray-centered variables. We present the initial integral here, then the leading order asymptotic approximation obtained by iterated steepest descent. Then we discuss the necessary transformations that lead us to an expression for the Hessian in the amplitude of the asymptotic expansion in terms of the dynamic quantities for the central rays of the Gaussian beams. That will complete the discussion of the true-amplitude adjustment from evaluation by the method of the steepest descent formula.

**Kinematic and dynamic variables of ray-centered coordinates**

Our discussion here is based on Červený [2001]. We begin by considering a ray defined by Cartesian ray theory with initial point \( x \) and passing through \( x' \) as in Figure 1. The initial...
Figure 1: Central ray of a Gaussian beam from \( x_0 \) through \( x' \), connected along an orthogonal vector \( q \) to an image point \( x \).

The polar angles define the initial slowness in the \( x \) direction of the ray from \( x_0 \) through \( x \) defines the rotation direction in 3D of the new coordinate system in 3D. Associated with that ray is an orthogonal ray-centered coordinate system, \((s, q_1, q_2)\). Figure 2 shows the initial orientation of the unit vectors \( (\hat{e}_1, \hat{e}_2) \) in an \((s, q_1, q_2)\) coordinate system. In these coordinates \( s \) is arc length along the central ray, as depicted in Figure 1, and \( q_1 \) and \( q_2 \) are the orthogonal coordinates to the central ray, also orthogonal to one another. While we need those angles for the theoretical derivation, we will not need them for the final evaluation of our dynamic quantities. Each of the central rays of Gaussian beams, as in Figure 1, has its own such triple of initial directions. In Figure 1, we depict the coordinates of the point \( x \) in terms of the ray-centered coordinates \((s, q_1, q_2)\) on the central ray of a Gaussian beam through \( x' \). In these coordinates, \( q_1 = q_2 = 0 \) on the central ray through \( x' \). Nearby central rays of Gaussian beams from the same point \( x_0 \) are defined by initial directions that are different from the initial direction of this central ray.

We need a notation for the velocity and its derivatives along the central ray through \( x' \):

\[
\begin{align*}
v_0(s) &= v(s, 0, 0), \\
v_{0,q} &= \frac{\partial v(s, 0, 0)}{\partial s} |_{q_1 = q_2 = 0}, \\
v_0,q &= \frac{\partial^2 v}{\partial q_1} |_{q_1 = q_2 = 0}, \\
v_0,q,q &= \frac{\partial^2 v}{\partial q_1 \partial q_2} |_{q_1 = q_2 = 0}.
\end{align*}
\]

The kinematic behavior of the central rays is determined by describing the propagation of the coordinates \( q_0 = (q_1, q_2) \) as a function of \( s \) along the central ray and the slownesses of the real traveltime \( \tau \) of asymptotic ray theory. In our application, we will need to move from initial slownesses \((p_1, p_2)\) of this central ray in the Cartesian coordinates to the initial ray-centered slownesses \( p_0 = (p_{01}, p_{02}) \) which are orthogonal to the central ray.

The kinematic equations in vector form are then given by

\[
\begin{align*}
\frac{dq}{ds} &= v_0 p, \\
\frac{dp}{ds} &= -\frac{1}{v_0} V q, \quad p(0) = p_0; \\
p_1 &= \frac{\partial \tau}{\partial q_1}, \\
p_2 &= \frac{\partial \tau}{\partial q_2}.
\end{align*}
\]

Here, \( V \) is the matrix defined in equation (1) and the vectors are vertical arrays of the \( q \)'s and \( p \)'s, respectively. The propagation of the transverse slownesses \((p_1, p_2)\) is determined. We determine the \( q \)'s of interest through the connection of the central ray to the image point.

We turn now to the dynamic equations of ray centered coordinates. These are equations for two \( 2 \times 2 \) matrices, \( Q \) and \( P \). These matrices are, in turn, related to the Hessian matrix of second derivatives of the traveltime with respect to \( q \) as follows:

\[
M = \begin{bmatrix} \frac{\partial^2 \tau}{\partial q_1 \partial q_2} \\ \frac{\partial^2 \tau}{\partial q_1 \partial q_2} \end{bmatrix}, \quad I, J = 1, 2 \quad M = PQ^{-1}.
\]

with \( Q \) and \( P \) satisfying the equations

\[
\frac{dQ}{ds} = v_0 P, \quad \frac{dP}{ds} = -\frac{1}{v_0} V Q.
\]

These are the same equations as the kinematic equations (2), except now for \( 2 \times 2 \) matrices. We refrain from stating initial conditions since we will define different solutions to these equations below by imposing different initial conditions. Furthermore, when we introduce Gaussian beams through complex-valued initial conditions for \( Q \) and \( P \), we will use \( T \) instead of \( \tau \) for the resulting complex traveltime. Then, \( \tau \) remains the real traveltime of classical asymptotic ray theory. However, the relation between the Hessian of \( T \) with respect to \( q \) and the matrices \( M, Q \) and \( P \) stated in equation (3) will remain the same.

Červený [2001] introduces two elementary solutions for the dynamic quantities, \( Q_1, P_1 \) and \( Q_2, P_2 \). These solutions satisfy the dynamic equation (4) and the initial conditions

\[
\begin{align*}
Q_1(x_0, x_0) &= I, \\
P_1(x_0, x_0) &= 0, \\
Q_2(x_0, x_0) &= 0, \\
P_2(x_0, x_0) &= I.
\end{align*}
\]

Now let us consider a complex-valued traveltime, \( T \), instead of the real traveltime \( \tau \) used above in equation (3). Thus, instead of equation (3), we write

\[
M_{cb} = \begin{bmatrix} \frac{\partial^2 T}{\partial q_1 \partial q_2} \\ \frac{\partial^2 T}{\partial q_1 \partial q_2} \end{bmatrix}, \quad I, J = 1, 2 \quad M_{cb} = P_{cb} Q_{cb}^{-1}.
\]
The pair of functions $Q(x', x_0) = Q_{gb}(x', x_0)$ and $P(x', x_0) = P_{gb}(x', x_0)$ are solutions of the dynamic equations (4) subject to the initial conditions proposed by Hill [2001]:

$$Q_{gb}(x_0, x_0) = \frac{\omega_0 w_0^2}{v_0(0)} I, \quad P_{gb}(x_0, x_0) = \frac{i}{v_0(0)} I,$$

(7)

$$Q_{gb}(x', x_0) = \frac{\omega_0 w_0^2}{v_0(0)} Q_1(x', x_0) + \frac{i}{v_0(0)} Q_2(x', x_0),$$

$$P_{gb}(x', x_0) = \frac{\omega_0 w_0^2}{v_0(0)} P_1(x', x_0) + \frac{i}{v_0(0)} P_2(x', x_0).$$

In the first line of this equation, $\omega_0$ is a reference frequency and $w_0$ is a length scale. At the reference frequency $\omega_0$, $w_0$ is the initial “standard deviation” of the Gaussian exponential that arises in the description of the Gaussian beam.

The product of Green’s functions and the integral to be analyzed

Any correlation-type migration imaging condition involves a product of Green’s functions from source and receiver to image point:

$$G^s(x, x_r, \omega)G^r(x, x_r, \omega) = \frac{-\omega^2 \omega_0^4}{4\pi^2 c^2(x_r) v^2(x_r)} I(x, x_r, x_r, \omega),$$

$$I(x, x_r, x_r, \omega) = \int_{D_s} \frac{dp_1}{p_{13}} \frac{dp_2}{p_{23}} \int_{D_r} \frac{dp_1}{p_1} \frac{dp_2}{p_2}$$

$$\cdot A^*_a(x'_s, x'_r) A^*_a(x'_s, x'_r) \exp\left\{-\omega \Psi(x'_s, x'_s, x'_r, x'_r)\right\},$$

$$\Psi(x'_s, x'_s, x'_s, x'_r) = -i[T(x'_s, x'_r) + T(x'_s, x'_r)]^*.$$  (8)

Here and below (‘) denotes complex conjugate. The variables $p_1', p_2', p_{13}, p_{23}$ are the initial Cartesian slownesses for the central rays of Gaussian beams from the source or receiver, respectively, each to their own point $x'_s$ or $x'_r$ equivalent to $x'_r$ in Figure 1. For a deconvolution-type integral formula, the exponent will be the same, but the amplitude will be different.

The integral $I$ in equation (8) is a prototype for the method of iterated steepest descent to be applied. Also, the $A$’s and $T$’s in equation (8) are

$$A_{gb}(x'_r, x_0) = \sqrt{\frac{v_r(x'_r)}{\det(Q_{gb}(x'_r))}},$$

(9)

$$T(x'_r, x_0) = \tau(s) + \frac{1}{2} q^T P_{gb} Q_{gb}^{-1} q, \quad \tau(s) = \int_0^s ds' \frac{v_0(s')}{v_0(0)}.$$  

In these equations, $v_0$ is the wave speed on the central ray of the Gaussian beam.

Hill [2001] proposes the change of variables

$$p_{1h} = p_{11} - p_{13}, \quad p_{2h} = p_{12} - p_{23},$$

$$p_{1m} = p_{11} + p_{13}, \quad p_{2m} = p_{12} + p_{23}.$$  (10)

The variables in $p_{1h}$ may be viewed as offset slowness vectors and the variables in $p_{2m}$ may be viewed as midpoint slownesses.

This change of variables produces an expressions for $I$ with the $(p_{1h}, p_{2m})$ integrals outermost. Hill then proposes that the method of steepest descent be applied to the two integrals in the variables $p_{1h}$ and $p_{2m}$. While acknowledging the approximate nature of doing this, he carries it out as a kinematic process, neglecting a complete amplitude and phase analysis. Therefore, he needs only to determine the saddle points in these two variables for any given choice of the other pair, $p_{1m}'$ and $p_{2m}'$. He then proposes to compute the integrals over $p_{1m}'$ and $p_{2m}'$ numerically.

We first identify the stationary point of the new integral. That point is now called a saddle point because the local structure of the surface $\Re[-\Psi]$ in one complex variable looks like a saddle. See Bleistein [1984] for an example. For the double integral in $p_{1h}'$, we assume that for each value of $p_{m}' = (p_{1m}', p_{2m}')$ there exists a “simple saddle point”

$$p_{h1}'(p_m) = (p_{h11}'(p_m), p_{h21}'(p_m), p_{h22}'(p_m)).$$  (11)

for which

$$\frac{\partial \Psi}{\partial p_{h1}} = \frac{\partial \Psi}{\partial p_{h2}} = 0, \quad p_h = p_{h1}'(p_m),$$

$$\det[\Psi] \neq 0, \quad \Psi = \left[\frac{\partial^2 \Psi}{\partial p_{h1} \partial p_{h2}}\right], \quad I, J = 1, 2, \quad p_h = p_{h1}'(p_m).$$  (12)

Under these assumptions, the iterated method of steepest descent leads to the following asymptotic formula for $I$ in equation (8).

$$I(x, x_r, x_r, \omega) \sim \pi \frac{\omega}{2} \int_{D_s} \frac{dp_{m1} dp_{m2}}{\sqrt{\det[\Psi]}} A^*_a(x'_s, x'_s) A^*_a(x'_r, x'_r)$$

$$\cdot \exp\{i\omega [T(x'_s, x'_r) + T(x'_s, x'_r)]^*\}, \quad p_h = p_{h1}'(p_m).$$  (13)

Analysis of the matrix $\Psi$

Now we need to express the Hessian matrix $\Psi$ of equation (12) into expressions containing the matrices $M$ of equation(3) for the source and receiver rays of the ray-centered variables for source and receiver. As a first step, we rewrite the matrix $\Psi$ of equation (12) in terms of the matrices of the separate traveltimes, by using equation (10) for the definition of the variables in $p_{h1}'$ in terms of the variables of $p_{1h}'$ and $p_{2m}'$. That leads to the following representation of the elements of $\Psi$.

$$\frac{\partial^2 \Psi}{\partial p_{h1} \partial p_{h2}} = -i \left[\frac{\partial^2 T_r(x'_s, x'_r)}{\partial p_{11}^2} \frac{\partial^2 T_r(x'_s, x'_r)}{\partial p_{21}^2} \right]^*$$

$$\frac{\partial^2 T_r(x'_s, x'_r)}{\partial p_{1j} \partial p_{2k}} = \frac{\partial^2 T_r(x'_s, x'_r)}{\partial p_{1j} \partial p_{2k}} + \frac{\partial^2 T_r(x'_s, x'_r)}{\partial p_{1k} \partial p_{2j}}.$$  (14)

We see now that in this equation the same transformations must be applied to the Cartesian derivatives of the source and receiver traveltimes to obtain the derivatives with respect to the kinematic variables, $q_{1s}$ or $q_{2s}$. Hence we can dispense with the subscripts $r$ and $s$ for now.

We first rotate the Cartesian coordinates to the angles $\beta_1$ and $\beta_2$ of Figure 2. The initial transverse slowness vector $(p_{01}, p_{02})$
in the ray-centered coordinate system for the central ray of a Gaussian beam is equal to zero when \( x' = x \). See Figure 1.

We can then transform each matrix \( T \) to derivatives with respect to \((p_1, p_2)\). That result is

\[
T = \Gamma^T T_0 \Gamma, \quad T_0 = [T_{\lambda\kappa}], \quad T_{\lambda\kappa} = \frac{\partial^2 T(x', x)}{\partial q_{\lambda\mu} \partial q_{\kappa\nu}}, \quad \lambda, \kappa = 1, 2,
\]

\[
\left[ \frac{\partial p_{\lambda0}}{\partial p_{\nu j}} \right] = \left[ \begin{array}{cc} \cos \beta_0 & \sin \beta_0 \\ \cos \beta_1 & \cos \beta_1 \\ -\sin \beta_1 & \cos \beta_1 \end{array} \right] = [\Gamma_{\lambda j}] = \Gamma, \quad \lambda, j = 1, 2.
\]  

(15)

In this equation, \( \Gamma \) is the transformation matrix between the initial transverse slownesses \((p_1, p_2)\) in the original Cartesian coordinate system; \((p_{10}, p_{20})\) are the initial slownesses in the rotated coordinate system of Figure 2. We will have a rotation such as this one for the source coordinates and for the receiver coordinates.

Next, we need the Jacobian of the transformation from the initial values of the slowness \((p_{10}, p_{20})\) to the Hessians in offset coordinates, \((q_1, q_2)\) on the central ray of the Gaussian beam in source or receiver coordinates. That transformation is

\[
T_0 = Q_2 T_q Q_2^T, \quad T_q = \left[ \frac{\partial^2 T}{\partial q_{\mu} \partial q_{\nu}} \right], \quad (16)
\]

\[
Q_2 = \left[ \frac{\partial q_{\mu}}{\partial p_{\lambda0}} \right], \quad \mu, \nu, \lambda = 1, 2.
\]

Finally, in this sequence of transformations, we need to write \( T_q \) in terms of the dynamic variables on the central ray. That equation turns out to be

\[
T_q = -i \omega_0 w_0^2 v_0 q^{-1} G^{-1}_{GB} Q^{-1}_{GB} Q_2^T, \quad q = 0. \quad (17)
\]

We now move back through the sequence of formulas for the various Hessians that we introduced above. We use equations (16) and (15) along with equation (17) for this purpose. We conclude that

\[
T = -i \omega_0 w_0^2 v_0 \Gamma^T Q_{GB}^{-1} Q_2 \Gamma. \quad (18)
\]

Equation (14) tells us that we must add two Hessians of the form defined by the last equation, (18), in order to obtain the Hessian \( \Psi \) for the complex travelt ime \( \Psi \) which is related to the two complex traveltimes from source and receiver by equation (8). The only difference in the two components is that one is for sources and the other is for receivers. We can accomplish that distinction by introducing subscripts \( s \) and \( r \) in the right hand side of equation (18). Thus we find that

\[
\Psi = i \omega_0 w_0^2 \left\{ \frac{1}{v_{0s}(0)} \Gamma^T Q_{GB}^{-1} Q_{rs} \Gamma_s + \frac{1}{v_{0r}(0)} \Gamma^T Q_{GB}^{-1} Q_{rs} \Gamma_r \right\}^s. \quad (19)
\]

Note here that \( \omega_0 \) denotes “reference frequency” and is the same for Hessians associated with the traveltimes from source and receiver.

The matrix \( \Psi \) in equation (19) requires that its two constituents in source and receiver variables “line up” appropriately. Although this is relatively rarer than the cases in which one or the other of the matrices in the sum are singular, some regularization of the right side in equation (19) is necessary before calculating the determinant. It is the determinant \( \Psi \) in equation (19) that must be computed for the integral \( I \) in equation (13)

The only matrices in equation (19) that require knowledge of the ray from the source to the image point and from the receiver to the image point are the \( \Gamma \)'s. We explain now why we do not need to know the rotation angles in those matrices explicitly. It is necessary to assume that the rays that we actually compute are sufficiently dense that the discrete sum over rays is a sufficiently accurate approximation of the continuous integral over rays to satisfy whatever numerical accuracy criterion we impose. That means we can think of the discrete sum as having the properties of the integral.

Figure 3 depicts a typical pair of rays for a saddle point in \( p_b \). Note that for this pair of rays, there is exponential decay of the contributing Gaussian beams at the image point because at least one (both in this case) of the \( q \)-vectors is nonzero. When both pairs of \( q \)'s are small, there will be linear error in the evaluation of the \( \Gamma \)'s if we evaluate them on the central rays that we do know as compared to their evaluation on the central rays (possibly not computed) that pass through the image point \( x \).

In fact, the subsequent integration in \( p_m \) has zero exponential decay only when the rays from source and receiver rays pass through the image point \( x \). So, the dominant region in the integration over \( p_m \) is where both rays pass close or through the image point. Consequently, we can compute the \( \Gamma \)'s on the respective central rays in our discrete sum; we know that ray pairs nearest to the image point dominate the sum and that the errors in computing the \( \Gamma \)'s along those rays are small compared to their values on the ray pair through the image point.

**Conclusions**

We have reported here on a method for completing Hill’s [2001] asymptotic analysis of Gaussian beam migration with an amplitude adjustment. This adjustment retains the true-amplitude of the Gaussian beam migration in 3D. Thus, the output yields an estimate of an angularly dependent reflection coefficient at a determinable incidence angle from the peak amplitude on the reflector. The details of this analysis are presented in Gray and Bleistein [2009] and Bleistein and Gray [2010].
REFERENCES


