Raytracing and Traveltime Calculations for Orthorhombic Anisotropic Media

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SUMMARY

Tilted orthorhombic anisotropy exists in sediments with fractures that are aligned because of the presence of geological stress. Accurate seismic depth imaging requires coping with this form of anisotropy. In this paper, we investigate anisotropic raytracing in tilted orthorhombic media for quasi-P wave simulation and migration. First, we apply an acoustic approximation to the elastic wave equation to obtain the quasi-P wave dispersion equation, and then we propose a Hamiltonian function to derive the ray equations. We have implemented our orthorhombic raytracing in the wavefront reconstruction algorithm to compute traveltimes for Kirchhoff depth migration.
Introduction

Accurate seismic depth imaging algorithms require a depth velocity model which closely resembles actual subsurface velocity field. At any given subsurface location, actual seismic wave propagation may not have the same speed in all propagation directions; velocity anisotropy has to be introduced to the model. Presently, the majority of anisotropy seismic imaging is based on transversely isotropic (TI) approach, assuming the wave propagation is isotropic in (usually) the bedding plane (Alkhalifah, T., and Tsvankin, I., 1995).

In sedimentary basins, the shale formations tend to be nearly parallel layered, and TI often represents the anisotropy well. However, when nonzero geological stress fields are present, fracture sets might be formed and afterwards be filled with other materials, giving rise to variable propagation velocity in the bedding planes. As the result, fractured bedding in an otherwise TI system can cause orthorhombic anisotropy. Orthorhombic symmetry exists under various situations, such as a combination of parallel cracks with transversely isotropy in the background media (Wild and Crampin, 1991; Schoenberg and Helbig, 1997), or two sets of orthogonal vertical fractures in an isotropic background (Tsvankin, 2001). Seismic methods for fracture detection trace back to the 1980’s (Crampin, 1985, Thomsen, 1988), and wave propagation in orthorhombic media has been discussed in a number of publications. (Brown et al. 1991, Tsvankin, 1997)

Orthorhombic wave propagation has been applied in reverse time migration (RTM) (Zhang et al., 2011), yielding better image focusing in production imaging projects. As the critical part of velocity tomography, orthorhombic raytracing is also been studied in several publications (Xie, Y, et al., 2011).

In this paper, we develop raytracing in orthorhombic media based on a Hamiltonian formulation (Farra and Madariaga, 1987); we investigate the effects of Thomsen parameters on our raytracer; and we confirm the raytracing results by comparing them with the propagator used in an orthorhombic reverse time migration (RTM).

Ray equations in orthorhombic media

Following the acoustic approximation in an orthorhombic system (Tsvankin, 1997), the relation of stiffness tensor $\sigma_{ij}$ and strain $\varepsilon_{ij}$ can be represented by the Thomsen parameters in the acoustic case:

$$
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33}
\end{bmatrix} = \rho(v_0)^2 \begin{bmatrix}
1 + 2\varepsilon_2 \\
(1 + 2\varepsilon_2) \sqrt{1 + 2\delta_3} \sqrt{1 + 2\delta_2} \\
\sqrt{1 + 2\delta_2}
\end{bmatrix} \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33}
\end{bmatrix}
$$

where $\rho$ is the density, assumed to be locally constant because it does not affect the propagation traveltime (phase term of Green’s function), $v_0$ is the velocity along $z$ direction, which is assumed to be the slowest propagation direction, $\varepsilon_1$, $\varepsilon_2$, $\delta_1$, $\delta_2$, $\delta_3$ are Thomsen parameters in orthorhombic system. $\varepsilon_1$, $\delta_1$ are defined in symmetry plane $[y, z]$, and $\varepsilon_2$, $\delta_2$ are defined in symmetry plane $[x, z]$. Without loss of generality, we assume $\varepsilon_1 > \varepsilon_2$. (Later, we consider the case where the anisotropy has nonzero tilt relative to the coordinate axes.)

Strain $\varepsilon_{ij}$ and stress $\sigma_{ij}$ in equation (1) can be replaced by displacement vector $\ddot{u}$ using the relations $\ddot{\varepsilon} = \ddot{\sigma} = \rho \ddot{u}$, and the plane wave phase velocity in direction $\ddot{\nu} = (n_x, n_y, n_z)$ can be calculated by solving for eigenvalue $\lambda$ in the linear equation

$$\begin{bmatrix}
\sigma_{11} - \lambda & \sigma_{12} \\
\sigma_{21} & \sigma_{22} - \lambda
\end{bmatrix} = \begin{bmatrix}e_{11}
\varepsilon_{22}
\varepsilon_{33}
\end{bmatrix}$$

The eigenvalues are then given by

$$\lambda = \frac{1}{2} \left( \sigma_{11} + \sigma_{22} + \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\varepsilon_2} \right)$$

$$\lambda = \frac{1}{2} \left( \sigma_{11} + \sigma_{22} - \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\varepsilon_2} \right)$$
Equation (2) yields a cubic equation for \( \lambda \) with constant term

\[
\text{Const} \equiv (AB - B^3E + 2BE^{1/2} + 2CDE^{1/2} - AD - BC)n_x^2n_y^2n_z^2
\]

where \( A = 1 + 2\varepsilon_x \), \( B = 1 + 2\varepsilon_y \), \( C = 1 + 2\delta_x \), \( D = 1 + 2\delta_y \), \( E = 1 + 2\delta_z \)

The three roots of cubic equation (2) represent one phase dispersion relation for quasi-P wave and two shear-wave relations. To simply the equation further, we assume that the medium is quasi acoustic, and one of the shear wave velocities vanishes, which yields the equation

\[
\sqrt{1 + 2\delta_x} = (\sqrt{1 + 2\delta_x} + \sqrt{1 + 2\delta_y} + \sqrt{1 + 2\delta_z})(1 + 2\varepsilon_z)
\]

Equation (2) now becomes a quadratic equation of \( \lambda \), with roots

\[
\lambda_\pm = \frac{1}{2} \left( (Bn_x^2 + An_x^2 + n_x^2) \pm \sqrt{(Bn_x^2 + An_x^2 + n_x^2) - 4[2Gn_y^2 + 2Fn_y^2 + A(B - AE)n_y^2]} \right)
\]

where \( F = \varepsilon_x - \delta_x \), \( G = \varepsilon_z - \delta_z \), and the definition of \( A, B, C, D, E \) are same as in (3), with the plus sign denoting the quasi-P dispersion relationship.

If unit vector \( \vec{n} \) is replaced by slowness vector \( \vec{p} \) in equation (5), we obtain

\[
\lambda_\pm(p) = s^2(\tilde{x}),
\]

where \( s \) is the phase slowness at location \( \tilde{x} \), and \( \tilde{p} = (p_x, p_y, p_z) \)

We next propose a Hamiltonian for raytracing:

\[
H(\tilde{x}, \tilde{p}) = \frac{1}{2} (\lambda_\pm(p) - s^2(\tilde{x})) = \frac{1}{2} \left( K + \sqrt{K^2 - 4L + A(B - AE)p_x^2p_y^2} \right) - 2s^2
\]

where \( K = (1 + 2\varepsilon_x)p_x^2 + (1 + 2\varepsilon_y)p_y^2 + (1 + 2\varepsilon_z)p_z^2 \), and \( L = 2((\varepsilon_x - \delta_x)p_x^2 + (\varepsilon_y - \delta_y)p_y^2 + (\varepsilon_z - \delta_z)p_z^2) \)

Orthorhombic kinematic raytracing equations system (8) can be established by integrating this Hamiltonian system:

\[
\begin{align*}
\frac{dx}{d\tau} &= \frac{dH(\tilde{x}, \tilde{p})}{dp_x} = [Bs^2 - 2Gp_z^2 - B[A - BE]p_z^2]p_x/M \\
\frac{dy}{d\tau} &= \frac{dH(\tilde{x}, \tilde{p})}{dp_y} = [As^2 - 2Fp_z^2 - B[A - BE]p_z^2]p_y/M \\
\frac{dz}{d\tau} &= \frac{dH(\tilde{x}, \tilde{p})}{dp_z} = s^2 - [2Gp_z^2 + Fp_z^2]p_z/M \\
\frac{dp_x}{d\tau} &= -\frac{dH(\tilde{x}, \tilde{p})}{dx} = s\sqrt{x} - [Rz^2 - p_x^2N - p_z^2p_y^2U]p_x/M \\
\frac{dp_y}{d\tau} &= -\frac{dH(\tilde{x}, \tilde{p})}{dy} = p_x \frac{dx}{d\tau} + p_y \frac{dy}{d\tau} + p_z \frac{dz}{d\tau}
\end{align*}
\]

where \( A, B, C, D, E, F, G, L, K \), and \( K \) are defined as before, \( M = 2s^2 - K \), \( R = p_y^2\nabla E_x + p_z^2\nabla E_z \), \( N = p_x^2\nabla G + p_y^2\nabla F \), \( U = \nabla \varepsilon_x A + B[\nabla \varepsilon_x - 2\nabla \varepsilon_y E - \nabla \varepsilon_z B] \).

Slowness vector \( \tilde{p} \) in (8) can be obtained by combining equation (5) and the relation \( \tilde{p} = \lambda_{\pm^{-1/2}}\tilde{n} \).

Raytracing in tilted orthorhombic media

The transformation system between the physical coordinates and the rotated coordinates, in which all three symmetry axes are aligned with coordinate axes, is defined by
\[
\begin{pmatrix}
\bar{x} \\
\bar{y} \\
\bar{z}
\end{pmatrix} =
\begin{pmatrix}
\cos\alpha & \sin\alpha & 0 \\
-\sin\alpha & \cos\alpha & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos\theta & 0 & -\sin\theta \\
0 & 1 & 0 \\
-\sin\phi & \cos\phi & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\]
(10)

Here \(\theta\) and \(\phi\) are defined by the rotation of the vertical axis at each spatial location, as we normally do for TTI media, and \(\alpha\) is the rotation angle of the elastic tensor in \([x, y]\) plane, which is usually identical to the angle between the crack orientation and (arbitrarily chosen) \(y\) axis. If \((x, y, z)\) are the spatial coordinates in physical coordinates, we denote as \((\bar{x}, \bar{y}, \bar{z})\) the coordinates of the same point in the rotated system, which now has the orthorhombic symmetry planes aligned with the coordinate axes denoted by overbars.

**Examples**

To test our orthorhombic ray tracer based on the derivation above, we embedded it into a wavefront reconstruction routine (Vinje et al., 1993), and used it to compute traveltimes for prestack Kirchhoff depth migration; then we compared the Kirchhoff migration impulse responses with those from TI Kirchhoff migration and orthorhombic finite difference reverse-time migration (RTM).

In the first example, we selected VTI anisotropy parameters, so we can use Kirchhoff migration with VTI raytracing to confirm our orthorhombic raytracing method in this degenerate case. We chose \(v_0 = 2000m/s\), \(\delta_1 = \delta_2 = 0.04\), \(\varepsilon_1 = \varepsilon_2 = 0.08\), \(\alpha = 0\). The orthorhombic impulse image should agree with a VTI image in a medium with parameters \(v_0 = 2000m/s\), \(\delta = 0.04\), \(\varepsilon = 0.08\). As shown as Fig 1, the two results agree with each other.

![Figure 1](image1.png)
*Figure 1: The Kirchhoff impulse migrations use orthorhombic raytracing (left) and VTI raytracing (right) to calculate times. The two images agree with each other.*

In the second example, we compare with an RTM impulse, which uses orthorhombic quasi-P propagation. For this example, the orthorhombic parameters are: \(v_0 = 2000m/s\), \(\varepsilon_1 = 0.32\), \(\varepsilon_2 = 0.08\), \(\delta_1 = \delta_2 = 0.04\). The result is shown in Fig 2. For both images, the impulse result in \([x, y]\) plane is an ellipse instead of a circle. The longer axis of the ellipse is \(y\) axis for \(\varepsilon_1\) is larger than \(\varepsilon_2\). Because \(\varepsilon_1\) is larger than \(\varepsilon_2\), the impulses in the crossline direction are flatter than along the inline direction, which means the velocity in the crossline direction is larger than in the inline direction.

![Figure 2](image2.png)
*Figure 2: The Kirchhoff impulse image (left) agree with RTM impulse image (right) in the same Orth. media.*
In the final example, we set rotation angle $\alpha = 120^\circ$ in addition to the anisotropy parameters used for example 2. As we expected, the Kirchhoff impulse response is an ellipse with a rotation angle 120 degrees, which is shown in Fig 3.

**Figure 3:** The Kirchhoff impulse image in the orthorhombic media with a rotation angle $\alpha = 120^\circ$.

**Conclusion**

We proposed a Hamiltonian approach for tilted orthorhombic raytracing, and we implemented this system to obtain traveltimes for prestack Kirchhoff depth migration. Image comparisons with previously calibrated VTI Kirchhoff and orthorhombic RTM impulse responses show good agreement.

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